Lecture Notes on Quantum Field Theory

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Contents

Preface and Acknowledgement

This note is base on the lecture of quantum field theory, taught by Prof. Amir-Kian Kashani-Poor in École Normale Supérieure. Thanks to his excellent teaching and clear explanation, this note can come the world. The author also thanks Zhong-Wu Chen for providing his note as a reference. When I become a teacher one day, I will use it to teach my students. It is not an easy job to write a lecture note on quantum field theory. Lots of details we should care about but no books can help you. (The structure of the lecture note is similar to the book [1]) This note is trying to show every detail in the lecture on quantum field theory. Though it is not perfect, it is a good guide for new beginners learning the quantum field theory.

0.1 Notation and convention

The metric convention is messy in theoretical physics. Here we adopt the convention that

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1}$$

In this lecture we just consider the quantum field theory in flat spacetime equipped with Minkowski metric. And we adopt the Einstein summation convention.

Chapter 1

Introduction

1.1 Background

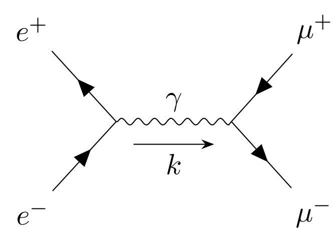
We needs some framework to describe elementary particles and their characters. QFT is quantum mechanical theory. What is quantum mechanical? The theory is based on the Hilbert space and the observable are the Hermitian operators acting on this space. However, we don't have like $H_0 = -\frac{\Delta}{2m}$ terms, which is non-relativistic. Relativistic QFT should satisfies two properties:

- Allow for particle creation/annihilation
- Lorentz invariance

Then comes to the two elementary questions:

- What quantum fields are allowed?
- What formalism we should use?
- How to extract physics?

A theory is said to be solve if we could solve the energy spectrum of the Hamiltonian. We will also consider the scattering problem using the perturbation theory. The computation method is called the Feynman diagram. The tree level diagram is as the following shown.



(1.1)

The loop diagram includes divergent integrals, which leads to the renormalization program.

1.2 Incorporating special relativity

We want to combine quantum mechanics and special relativity, which means that the Lorentz group must act on the Hilbert space. This guides us to study the representation theory, especially the representation of Lorentz group.

1.3 Incorporating the creation and annihilation of particles

1.3.1 Fock space

We denote $\mathscr{H}_n$ as the n-particle Hilbert space.

Example 1.1. *If we consider Free boson of a given mass m. The Hilbert space series is*

$$\begin{aligned}
&\mathscr{H}_0 = \mathbb{C} = \langle |0\rangle \rangle \\
&\mathscr{H}_1 = \begin{cases} L_2(\mathbb{R}^3) \quad \textit{normalizable} \\ \langle |p^\mu\rangle \, |p^2 = m^2, p^0 > 0\rangle \quad \textit{non-normalizable} \end{cases} \\
&\mathscr{H}_2 = \begin{cases} S\left(L_2(\mathbb{R}^3 \otimes L_2(\mathbb{R}^3))\right) \\ \langle |p_1^\mu, p_2^\mu\rangle \, |p_i^2 = m^2, p_i^0 > 0\rangle \end{cases}
\end{aligned} \tag{1.2}$$

where S denotes the symmetrization. The Fock space is the direct sum of these spaces

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n \tag{1.3}$$

Definition 1.1. *We define the inner product of this space, $(\cdot,\cdot) : \mathscr{F} \times \mathscr{F} \to \mathbb{C}$, which satisfies*

- $(\cdot,\cdot)|_{\mathscr{H}_n \times \mathscr{H}_n} := (\cdot,\cdot)|_{\mathscr{H}_n}$
- $(\cdot,\cdot)|_{\mathscr{H}_m \times \mathscr{H}_n} = 0 \quad \textit{if} \quad m \neq n$

Normalization of the inner product:

$$\begin{aligned}
&\mathscr{H}_0 : \langle 0|0\rangle = 1 \\
&\mathscr{H}_1 : \langle p|k\rangle = 2\omega_p (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k}),\ \omega_p = \sqrt{m^2 + \vec{p}^2} = p^0 \\
&\ldots \\
&\mathscr{H}_n = \langle k_1, \ldots, k_n | p_1, \ldots, p_n\rangle = \prod_{i=1}^{n} 2\omega_{k_i} \sum_{\delta \in \mathbb{S}_n} \prod_{j=1} (2\pi)^3 \delta^{(3)}(\vec{p}_j - \vec{k}_{\delta(i)})
\end{aligned} \tag{1.4}$$

For example,

$$\langle p_1, p_2 | k_1, k_2\rangle = 2\omega_{p_1} 2\omega_{p_2} \left[(2\pi)^3 \delta^{(3)}(\vec{k_1} - \vec{p_1})(2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{p}_2) + (2\pi)^3 \delta^{(3)}(\vec{k_1} - \vec{p_2})(2\pi)^3 \delta^{(3)}(\vec{k}_2 - \vec{p}_1) \right] \tag{1.5}$$

1.3.2 Creation and annihilation operators

If we fix the mass m of the particle, for any $\vec{p} \in \mathbb{R}^3$, we define

$$a_p^\dagger |p_1, \ldots, p_n\rangle = \frac{1}{\sqrt{2\omega_p}} |p, p_1, \ldots, p_n\rangle \tag{1.6}$$

Remark 1.1. *Here we don't use the vector notation $\vec{p}$ because we require the momentum are on-shell, which means that $p^0 = \sqrt{m^2 + \vec{p}^2}$ and $\vec{p}$ fixes the four vector p.*

As
$$|p_1, p_2, \ldots, p_n\rangle = |p_2, p_1, \ldots, p_n\rangle \ , \tag{1.7}$$
we have
$$[a^\dagger_{p_1}, a^\dagger_{p_2}] = 0 \,. \tag{1.8}$$

Properties 1.1. *Properties of the adjoint operator a_p: Let $|\psi\rangle \in \mathscr{H}_m$, $|\phi\rangle \in \mathscr{H}_n$,*
$$\Big(a_p a^\dagger_k \,|\psi\rangle \,, |\phi\rangle\Big) = \Big(a^\dagger_k \,|\psi\rangle \,, a^\dagger_p \,|\phi\rangle\Big) \ . \tag{1.9}$$
If we require 1.9 is non zero, we can see that $m = n$. If it is zero, we can see that $a_p a^\dagger_k \mathscr{H}_m \perp \mathscr{H}_n$. These lead to
$$a_p a^\dagger_k\Big|_{\mathscr{H}_m} : \mathscr{H}_m \to \mathscr{H}_m \ \Rightarrow \quad a_p : \mathscr{H}_m \to \mathscr{H}_{m-1} \,. \tag{1.10}$$

But we have to be careful about the case when $m = 0$. For a state $|\psi\rangle \in \mathscr{F}$, $a^\dagger_k \,|\psi\rangle \in \otimes_{n=1}^{\infty} \mathscr{H}_n$, which means that
$$\Big(|0\rangle \,, a^\dagger_k \,|\psi\rangle\Big) = 0 = (a_k \,|0\rangle \,, |\psi\rangle) \quad \Rightarrow \quad a_k \,|0\rangle \perp \mathscr{F} \tag{1.11}$$
The only possible case is
$$a_k \,|0\rangle = 0 \,. \tag{1.12}$$
Along the same line of reasoning
$$a_k \,|k_1, \ldots, k_n\rangle = \sum_{i=1}^{n} \sqrt{2\omega_{k_i}} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_i) \,|k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n\rangle \ . \tag{1.13}$$

We need to determine the commutator $[a^\dagger_k, a_p]$. Noticing that
$$a^\dagger_p a_k \,|k_1, \ldots, k_n\rangle = \sum_{i=1}^{n} \sqrt{\frac{2\omega_{k_i}}{2\omega_p}} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_i) \,|p, k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n\rangle \,, \tag{1.14}$$
$$a_k a^\dagger_p \,|k_1, \ldots, k_n\rangle = \sum_{i=1} \sqrt{\frac{2\omega_{k_i}}{2\omega_p}} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_i) \,|p, k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n\rangle \tag{1.15}$$
$$+ \sqrt{\frac{2\omega_p}{2\omega_p}} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) \,|k_1, \ldots, k_n\rangle \ . \tag{1.16}$$
This leads that
$$[a^\dagger_p, a_k] \,|k_1, \ldots, k_n\rangle = -(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k}) \,|k_1, \ldots, k_n\rangle \ . \tag{1.17}$$
For arbitrary basis $|k_1, \ldots, k_n\rangle \in \mathscr{F}$ the equation holds, the only possibility is
$$[a_k, a^\dagger_p] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) \,. \tag{1.18}$$

To summarize, the creation and annihilation operator in the Fock space is

$$\text{Creation:} \quad a_p^\dagger|_{\mathscr{H}_m} : \mathscr{H}_m \to \mathscr{H}_{m+1}\,, \tag{1.19}$$

$$\text{Annihilation:} \quad a_p|_{\mathscr{H}_m} : \mathscr{H}_m \to \mathscr{H}_{m-1}\,. \tag{1.20}$$

They satisfy the commutation relation

$$[a_p^\dagger, a_k^\dagger] = [a_p, a_k] = 0, \quad [a_p, a_k^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{k})\,. \tag{1.21}$$

Remark 1.2. *We need to be careful about the normalization factor for symmetrization, e.g.*

$$|p_1, p_2\rangle = \frac{1}{\sqrt{2!}} \left(|p_1\rangle \otimes |p_2\rangle + |p_2\rangle \otimes |p_1\rangle \right)\,. \tag{1.22}$$

1.3.3 Identity operator on $\mathscr{H}_1$

The identity operator is

$$\mathbb{1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |p\rangle \langle p|\,. \tag{1.23}$$

The action of the identity operator is

$$\mathbb{1}\,|k\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |p\rangle \langle p|k\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |p\rangle\, (2\pi)^3 2\omega_p \delta^{(3)}(\vec{p} - \vec{k}) = |k\rangle\,. \tag{1.24}$$

We need to be careful about the Lorentz invariance of each ingredients. First,

Theorem 1.1. *The measure part $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}$ is Lorentz invariant.*

Proof: Recall the property of the delta function:

$$\delta(f(x)) = \sum_{f(x_0)=0} \frac{1}{|f'(x_0)|} \delta(x - x_0)\,. \tag{1.25}$$

We could see that factor of the the on-shell momentum p could be rewritten as

$$\frac{1}{|2p^0|} \left[\delta(p^0 - \sqrt{\vec{p}^2 + m^2}) + \delta(p^0 + \sqrt{\vec{p}^2 + m^2}) \right]\,. \tag{1.26}$$

Rewrite the expression

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} = \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta\left((p^0)^2 - \vec{p}^2 - m^2 \right) \theta(p^0)\,. \tag{1.27}$$

The factor d^4p is Lorentz invariant as $\det \Lambda = \pm 1$ for $\Lambda \in O(1,3)$. If we restrict $\Lambda \in SO(1,3)$, $\det \Lambda = 1$. $\theta(p^0)$ is invariant under the orthochorous Lorentz group $O^\uparrow(1,3)$, which proceeds the direction of time. So the it is indeed Lorentz invariant.

1.3.4 Constructing operator out of $a_p, a_p^\dagger$

The Hamiltonian is

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p \,, \tag{1.28}$$

such that

$$H \left| k_1, \ldots, k_n \right\rangle = \left(\prod_i^n \sqrt{2\omega_{k_i}} \right) \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p a_{k_1}^\dagger \ldots a_{k_n}^\dagger \left| 0 \right\rangle \,. \tag{1.29}$$

To simplify the expression, we need to exchange the position of a_p and $a_{k_1}^\dagger$. Since $a_p a_{k_1}^\dagger = a_{k_1}^\dagger a_p + (2\pi)^3 \delta^{(3)} \left(\vec{k_1} - \vec{p} \right)$, the expression then becomes

$$\left(\prod_i^n \sqrt{2\omega_{k_i}} \right) \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_{k_1}^\dagger a_p \ldots a_{k_n}^\dagger \left| 0 \right\rangle + \omega_{k_1} a_{k_1}^\dagger \ldots a_{k_n}^\dagger \left| 0 \right\rangle \,. \tag{1.30}$$

Repeating this procedure until the annihilation operator directly acts on the vacuum, the expression reduces to

$$\left(\sum_i \omega_{k_i} \right) \left| k_1, \ldots, k_n \right\rangle \,. \tag{1.31}$$

However, we need to emphasize that quantum field theory should be Lorentz invariant. We should check that the definition of the Hamiltonian meets this condition. This leads to the introduction of the quantum fields.

1.4 Quantum field

In this section, we would introduce the building blocks for Lorentz invariant theories. The free scalar field at a fixed time could be written as

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) \,, \tag{1.32}$$

where the subscript 0 represents the free field.

1.4.1 Interpretation of $\phi_0(\vec{x})$

$$\begin{aligned} \left\langle k \right| \phi_0(\vec{x}) \left| 0 \right\rangle &= \left\langle k \right| \int \frac{d^3p}{(2\pi^3)} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) \left| 0 \right\rangle \\ &= \left\langle k \right| \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \left| 0 \right\rangle = \left\langle k \right| \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \frac{1}{\sqrt{2\omega_p}} \frac{1}{\sqrt{2\omega_p}} \left| p \right\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \frac{1}{2\omega_p} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) = e^{-i\vec{k}\cdot\vec{x}} \propto \left\langle k | x \right\rangle \,, \end{aligned} \tag{1.33}$$

which means that $\phi_0(\vec{x}) \left| 0 \right\rangle$ can be interpreted as generating a particle at the position $\vec{x}$: $\phi_0(\vec{x}) \left| 0 \right\rangle \sim \left| x \right\rangle$.

Properties 1.2. *(a)* $\phi_0(\vec{x}) \left| 0 \right\rangle \in \mathscr{H}_1$.

1.4.2 Time dependence

The Hamiltonian H is the time evolution of operator

$$i[H, \mathscr{O}] = \partial_t \mathscr{O} \quad \text{for any operator } \mathscr{O} \,. \tag{1.34}$$

The solution of this operator differential equation is

$$\mathscr{O}(t) = e^{iHt}\mathscr{O}(0)e^{-iHt} \,. \tag{1.35}$$

Note that $\partial_t H = i[H, H] = 0$, hence H is time-independent.

$$H = \int \frac{d^3p}{(2\pi)^3}\left(\omega_p a_p^\dagger(0)a_p(0) + V(0)\right) = e^{-iHt}He^{iHt} = \int \frac{d^3p}{(2\pi)^3}\left(\omega_p a_p^\dagger(t)a_p(t) + V(t)\right) \tag{1.36}$$

which means that we could evaluate the a_p and $a_p^\dagger$ at any time t. The system has time-reversal symmetry. There is no special moment, which corresponding to the conservation of the energy.

Time dependence of $a_p, a_p^\dagger$ for free fields

$$\begin{aligned}
\partial_t a_k(t) = i[H_0(t), a_k(t)] &= i\int \frac{d^3p}{(2\pi)^3}\left[\omega_p a_p^\dagger(t)a_p(t), a_k(t)\right] \\
&= i\int \frac{d^3p}{(2\pi)^3} - (2\pi)^3\delta^{(3)}(\vec{k}-\vec{p})a_p(t) \\
&= -i\omega_k a_k(t) \,.
\end{aligned} \tag{1.37}$$

Again, the subscript 0 means the free Hamiltonian with no interaction and we have already used the commutation property to simplify the equation. The solution of this differential equation is direct

$$a_k(t) = a_k(0)e^{-i\omega_k t} \,. \tag{1.38}$$

Similarly,

$$a_k^\dagger(t) = e^{i\omega_k t}a_k^\dagger(0) \,. \tag{1.39}$$

Time dependence of free quantum field

$$\phi_0(0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(a_p(0)e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger(0)e^{-i\vec{p}\cdot\vec{x}}\right) \,. \tag{1.40}$$

$$\Rightarrow \quad \phi_0(t, \vec{x}) = e^{iHt}\phi_0(\vec{x})e^{-iHt} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p(0)e^{-ip\cdot x} + a_p(0)^\dagger e^{ip\cdot x}\right) \,, \tag{1.41}$$

where we use that convention that $p \cdot x = p^\mu x_\mu$, the equation (1.38, 1.39) and $p^0 = \omega_p$. Note that

$$\partial_\mu\partial^\mu\phi_0(x) := \Box\phi_0(x) = -\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{\sqrt{2\omega_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right) = -m^2\phi_0(x) \,. \tag{1.42}$$

The equation holds if the momentum is on-shell. We conclude that the free scalar field satisfies the Klein-Gordan equation.

Free field vs. interacting field

For more complicated Hamiltonian, we cannot solve for $a_p(t)$ analytically. If we set $a_p(t)$ and $a_p^\dagger(t)$, the scalar field could be written as

$$\phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[a_p(t)e^{-ip\cdot x} + a_p^\dagger(t)e^{ip\cdot x}\right] . \tag{1.43}$$

Here we define the $a_p(t)$ as

$$e^{iHt}a_p(0)e^{-iHt} = a_p(t)e^{-ip^0t} , \tag{1.44}$$

This seems a little bit tricky, but only this definition is consistent with the formalism $e^{iHt}\phi(0,\vec{x})e^{-iHt} = \phi(t,\vec{x})$. At equal time, the commutation relation is

$$\begin{aligned}[a_p(t), a_k^\dagger(t)] &= \left[e^{ip^0t}e^{iHt}a_p(0)e^{-iHt}, e^{-ik^0t}e^{iHt}a_k^\dagger(0)e^{-iHt}\right] = e^{i(p^0-k^0)t}e^{iHt}[a_p(0), a_k^\dagger(0)]e^{-iHt} \\ &= (2\pi)^3\delta^{(3)}(\vec{p}-\vec{k}) .\end{aligned} \tag{1.45}$$

where we use the on-shell condition (the spatial momentum could determine the time component of p^μ.).

1.4.3 Commutation relation of quantum fields

We will argue that $\phi(t,\vec{x})$ and $\Pi(t,\vec{x}) = \partial_t\phi(t,\vec{x})$ are canonically conjugate operators. At equal time, consider the commutator with $x = (t,\vec{x}), y = (t,\vec{y})$,

$$\begin{aligned}[\phi(t,\vec{x}), \phi(t,\vec{y})] = \int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_k}}\times \\ \left[a_k(t)e^{-ikx} + a_k^\dagger(t)e^{ikx}, a_p(t)e^{-ipx} + a_p^\dagger(t)e^{ipx}\right] .\end{aligned} \tag{1.46}$$

Evaluate the commutator, the nontrivial term is

$$e^{-i(k^0-p^0)t+i\vec{k}\cdot\vec{x}-i\vec{p}\cdot\vec{y}}[a_k(t), a_p^\dagger(t)] + e^{i(k^0-p^0)t-i\vec{k}\cdot\vec{x}+i\vec{p}\cdot\vec{y}}[a_k^\dagger(t), a_p(t)] , \tag{1.47}$$

because

$$[a_k(t), a_p^\dagger(t)] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{p}),\ [a_k^\dagger(t), a_p(t)] = -(2\pi)^3\delta^{(3)}(\vec{k}-\vec{p}) . \tag{1.48}$$

Substitute this term into the equation (1.46) is

$$[\phi(t,\vec{x}), \phi(t,\vec{y})] = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left[e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\right] . \tag{1.49}$$

Noticing that $\omega_k = \omega_{-k}$, the integral is integrated in the whole region of $\vec{k}$, if we consider the variable transform $\vec{k} \to -\vec{k}$, the integration is the same,

$$\Rightarrow\ [\phi(t,\vec{x}), \phi(t,\vec{y})] = 0 . \tag{1.50}$$

$$\Pi(t,\vec{x}) = \partial_t\phi(t,\vec{x}) = i[H, \phi(t,\vec{x})] = i[H_0 + V(\phi), \phi(t,\vec{x})] = i[H_0, \phi(t,\vec{x})]. \tag{1.51}$$

Consider the free field and using $i[H_0, a_k(0)] = -\omega_k a_k(0)$,

$$[\phi(t,\vec{x}), \Pi(t,\vec{y})] = -i \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sqrt{\frac{\omega_k}{2}} \left[a_p(t) e^{-ipx} + a_p^\dagger(t) e^{ipx}, a_k(t) e^{-ikx} - a_k^\dagger(t) e^{ikx} \right] . \quad (1.52)$$

Evaluate the commutator, the equation reduces to

$$[\phi(t,\vec{x}), \Pi(t,\vec{y})] = -\frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left(-e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right) = i \int \frac{d^3p}{(1\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = i\delta^{(3)}(\vec{x}-\vec{y}) . \quad (1.53)$$

Similar calculation shows $[\Pi(t,\vec{x}), \Pi(t,\vec{y})] = 0$.

Chapter 2

Classical Field Theory

2.1 From discrete case to fields

2.2 Noether theorem

Continuous symmetries of the Lagrangian leads to the conserved current.

Example 2.1. *Given the Lagrangian of the complex scalar field*

$$\mathscr{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 |\phi|^2 \,. \tag{2.1}$$

The complex scalar field has two independent degrees of freedom. There is two way to consider it. One way is to write the complex scalar field as $\phi = \phi_1 + i\phi_2$, where ϕ_1, ϕ_2 are both real scalar field. The other way is that ϕ and its conjugate ϕ^* are independent scalar field. The equation of motion derived by Euler-Lagrange equation is

$$(\Box + m^2)\phi = 0 \,. \tag{2.2}$$

$\mathscr{L}$ is obviously invariant under the global symmetry transformation

$$\phi \to e^{i\alpha}\phi \quad \phi^* \to e^{-i\alpha}\phi^* \,. \tag{2.3}$$

Such a symmetry is called $U(1)$ symmetry.

Noether theorem: Assume $\mathscr{L}(\phi_1, \ldots, \phi_n, \partial_\mu \phi_1, \ldots, \partial_\mu \phi_n)$ is invariant under the transformation

$$\phi_i(x) \to \tilde{\phi}_i(\alpha) := \phi_i(x; \alpha) \,, \tag{2.4}$$

where α is independent of x (global symmetry?). The Lagrangian is invariant under the transformation, which means that

$$\frac{\partial \mathscr{L}(\tilde{\phi}(\alpha), \partial_\mu \tilde{\phi}(\alpha))}{\partial \alpha} = 0 \,. \tag{2.5}$$

At the same time.,

$$\frac{\partial \mathscr{L}(\tilde{\phi}(\alpha), \partial_\mu \tilde{\phi}(\alpha))}{\partial \alpha} = \sum_n \left[\frac{\partial \mathscr{L}}{\partial \tilde{\phi}_n} \frac{\partial \tilde{\phi}_n}{\partial \alpha} + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \tilde{\phi}_n)} \frac{\partial(\partial_\mu \tilde{\phi}_n)}{\partial \alpha} \right] \,. \tag{2.6}$$

Rewrite the second term of the right hand side of the equation as

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \tilde{\phi}_n)} \frac{\partial(\partial_\mu \tilde{\phi}_n)}{\partial \alpha} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \tilde{\phi}_n)} \frac{\partial \tilde{\phi}_n}{\partial \alpha} \right) - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \tilde{\phi}_n)} \frac{\partial \tilde{\phi}_n}{\partial \alpha} \,. \tag{2.7}$$

Put out the common terms then the equation becomes

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \tilde{\phi}_n)} \frac{\partial(\partial_\mu \tilde{\phi}_n)}{\partial \alpha} &= \sum_n \left[\frac{\partial \mathscr{L}}{\partial \tilde{\phi}_n} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \tilde{\phi}_n)} \right] \frac{\partial \tilde{\phi}_n}{\partial \alpha} + \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \tilde{\phi}_n} \frac{\partial \tilde{\phi}_n}{\partial \alpha} \right) \\ &= \sum_n \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \tilde{\phi}_n} \frac{\partial \tilde{\phi}_n}{\partial \alpha} \right) , \end{aligned} \tag{2.8}$$

where we have already assumed that all fields satisfy the equation of motion at the second equal sign. The Noether conserved current is

$$\Rightarrow J^\mu = \sum_n \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \tilde{\phi}_n} \frac{\partial \tilde{\phi}_n}{\partial \alpha} \right) . \tag{2.9}$$

By invariance on α, evaluate at $\alpha = 0$ ($\phi(\vec{x}, 0) = \phi(x)$),

$$J^\mu = \sum_n \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta_\alpha \phi_n ,\; \delta_\alpha \phi_n = \left. \frac{\partial \tilde{\phi}_n}{\partial \alpha} \right|_{\alpha=0} . \tag{2.10}$$

Obviously $\partial_\mu J^\mu = 0$, which is the requirement of the conserved current. As the associate charge,

$$Q = \int d^3x J^0 \tag{2.11}$$

is conserved since

$$\partial_t Q = \int d^3x \partial_t J^0 = -\int d^3x \partial_i J^i = \text{surface integral} = 0 \,. \tag{2.12}$$

Example 2.2. *Take the* $\mathscr{L}$ *given in (2.1), and we define* $\tilde{\phi}(\alpha) = e^{-i\alpha}\phi$, $\tilde{\phi}^* = e^{i\alpha}\phi^*$. *The conserved current associated to the symmetry transformation is*

$$J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta_\alpha \phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^*)} \delta_\alpha \phi^* = \partial^\mu \phi^* (-i\phi) + \partial^\mu \phi (i\phi^*) = -i(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) \,. \tag{2.13}$$

One can verify that $\partial_\mu J^\mu = 0$.

2.3 The Maxwell Lagrangian and gauge invariance

Recall that $\vec{E}$ and $\vec{B}$, the electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is gauge field and $F_{\mu\nu}$ is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \epsilon, \tag{2.14}$$

where ϵ is arbitrary function of spacetime coordinate.

2.4 Distinction between gauge transformation and (global) symmetry transformation

The Lagrangian of the electromagnetic field is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \tag{2.15}$$

The Euler-Lagrange equation is

$$\partial_\mu F^{\mu\nu} = 0, \tag{2.16}$$

which is the vacuum Maxwell equation. If we couple electromagnetic field to external sources, the equation of motion will be

$$\partial_\mu F^{\mu\nu} = J^\nu \,. \tag{2.17}$$

Example 2.3. *The external sources associated to a static charge of strength e at the origin is*

$$J^\mu(x) = \begin{cases} \mu = 0, & \rho(x) = e\delta^{(3)}(\vec{x}), \\ \mu = i, & 0\,. \end{cases} \tag{2.18}$$

The corresponding Lagrangian associated with sources is

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu \,. \tag{2.19}$$

Not the question raises, what if we do the gauge transformation now? Is the Lagrangian still invariant?

$$\mathscr{L}' = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - (A_\mu + \partial_\mu \epsilon) J^\mu = \mathscr{L} - \partial_\mu(\epsilon J^\mu) + \epsilon \partial_\mu J^\mu \,. \tag{2.20}$$

The second term of the right hand side of the equation is the total derivative, which will be the boundary term after integration, so it could be ignored. If we impose the conserved condition $\partial_\mu J^\mu = 0$, the Lagrangian is invariant under the gauge transformation. In other words, if we require $\mathscr{L}$ is invariant under gauge transformation, it is equivalent to requires the coupling to conserved currents. Furthermore,

$$\partial_\mu F^{\mu\nu} = J^\mu \Rightarrow \partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu J^\nu = 0\,. \tag{2.21}$$

Considering conserved current also requires the equation of motion. Coupling electromagnetic field to a dynamic source, we require $\partial_\mu J^\mu = 0$, which we can say that it is on shell. It is not good enough, we will require gauge invariance.

2.5 Feynman rules in classical field theory

The goal of the section is to use Feynman rules (graphical method) to solve the equation of motion.

2.5.1 Ex: electromagnetic field with static source at the origin

Impose the Lorenz gauge $\partial_\mu A^\mu = 0$,

$$J^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \Box A^\nu \,, \tag{2.22}$$

using the component form,

$$\Box A_0 = e\delta^3(x), \quad \Box A_i = 0 \to A_i = 0\,. \tag{2.23}$$

The gauge condition requires $\partial_0 A^0 = 0$. Solve the differential equation through Fourier transformation method, setting

$$A_0(x) = \int \frac{d^3k}{(2\pi)^3} \tilde{A}_0(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \,, \tag{2.24}$$

which is the static Ansatz due to gauge condition.

$$\Box A_0(x) = \int \frac{d^3k}{(2\pi)^3} \Box_x \left(\tilde{A}_0(\vec{k}) e^{i\vec{k}\cdot\vec{x}}\right) = \int \frac{d^3k}{(2\pi)^3} k^2 \left(\tilde{A}_0(\vec{k}) e^{i\vec{k}\cdot\vec{x}}\right) = e \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \Rightarrow \tilde{A}_0(\vec{k}) = \frac{e}{\vec{k}^2}\,. \tag{2.25}$$

Inverse Fourier transform gives

$$A_0(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e}{\vec{k}^2} e^{i\vec{k}\cdot\vec{x}} = \frac{e}{4\pi}\frac{1}{r}\,. \tag{2.26}$$

2.5.2 General source: first Green function method

Inhomogeneous PDE for a δ-function is

$$\mathfrak{D}_x \phi = J, \quad \mathfrak{D}_x \Pi(x,y) = \delta^{(4)}(x-y)\,, \tag{2.27}$$

where Π is called the Green function. Then based on the PDE theory, for arbitrary source, the solution is $\phi(x) = \int d^4y \Pi(x,y) J(y)$,

$$\mathfrak{D}_x \phi(x) = \int d^4y \mathfrak{D}_x \Pi(x,y) J(y) = \int d^4y \delta^{(4)}(x-y) J(y) = J(x)\,. \tag{2.28}$$

Of course the Green function will be fixed by the boundary condition. Back to our problems, the differential operator is $\mathfrak{D}_x = -\Box$, based on the discussion above,

$$-\Box \Pi(x,y) = \delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \quad \Rightarrow \Pi(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{ik(x-y)}\,, \tag{2.29}$$

where the notation $k(x-y) := k^\mu(x_\mu - y_\mu)$.

2.5.3 More complicated green function $\mathfrak{D}_x$ determining solutions

Consider the Lagrangian,

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{3}\lambda \phi^3 + J\phi\,. \tag{2.30}$$

The equation of motion is

$$\Box \phi - (\lambda \phi^2 + J) = 0\,. \tag{2.31}$$

Suppose λ is small parameter, the Ansatz is $\phi = \sum_{n=0}^{+\infty} \lambda^n \phi_n$, satisfying

$$\Box\left(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \ldots\right) - \lambda\left(\phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \ldots\right)^2 - J = 0\,. \tag{2.32}$$

Reorganize the expression, we have

$$(\Box\phi_0 - J) + \lambda\left(\Box\phi_1 - \lambda\phi_0^2\right) + \lambda^2\left(\Box\phi_2 - 2\phi_0\phi_1\right) + \ldots = 0 \tag{2.33}$$

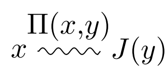
The solution for the coefficients are

$$\phi_0(x) = \int d^4y \Pi(x,y) J(y), \tag{2.34}$$

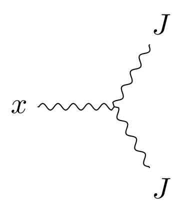
$$\phi_1(x) = -\int d^4y \Pi(x,y) \phi_0^2(y) \tag{2.35}$$

$$\phi_2(x) = -\int d^4y \Pi(x,y) 2\phi_0\phi_1\,, \tag{2.36}$$

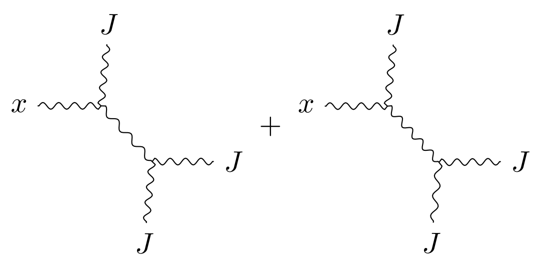
or diagrammatically,

$\phi_0 =$ 

(2.37)

$\phi_1 =$ 

(2.38)

$\phi_2 =$ 

(2.39)

The trivalent vertex comes from the interaction ϕ^3. Perturbation theory is governed by Feynman rules. The orders of contribution equal the number of vertices.

Remark 2.1. *Only tree level graphs occur, i.e no loops. In fact, the loop diagram comes from the quantum effect.*

2.6 From classical field theory to quantum field theory

The canonical quantization procedure is to replace the Poisson bracket $\{\cdot,\cdot\}$ by the commutator $\frac{1}{i}[\cdot,\cdot]$. However, ordering issues have to be dealt with. A corollary from Noether theorem is that for symmetries such that $\phi_n(x;\epsilon) = \mathscr{F}(\phi_n(x),\epsilon)$, i.e. the transformation doesn't include Π dependence.

Corollary 2.1.

$$[Q, \phi_n(x)] = -i\delta_\epsilon\phi_n(x) \quad \textit{where} \quad \delta_\epsilon\phi_n(x) = \left.\frac{\partial\phi_n(x)}{\partial\epsilon}\right|_{\epsilon=0} . \tag{2.40}$$

i.e. Q is the infinitesimal generator of the symmetry.

Proof:

$$\begin{aligned} [Q, \phi_m(t,\vec{x})] &= \left[\int d^3y J^0(t,\vec{y}), \phi_m(t,\vec{x})\right] \\ &= \left[\int d^3y \sum_n \frac{\partial\mathscr{L}}{\partial(\partial_0\phi_n)}\delta_\epsilon\phi_n(t,\vec{y}), \phi_m(t,\vec{x})\right] . \end{aligned} \tag{2.41}$$

By assumption, $\delta_\epsilon\phi_n(t,\vec{y})$ doesn't include $\Pi_k(t,\vec{y})$, so the expression is

$$\sum_n \int d^3y \, [\Pi_n(t,\vec{y}), \phi_m(t,\vec{x})] \, \delta_\epsilon\phi_n(t,\vec{y}) = -i\delta_\epsilon\phi_m(\vec{x}) . \tag{2.42}$$

where we have already used the commutation relation between Π and ϕ.

Chapter 3

In and out states and the S-matrix

3.1 Scattering

The scattering picture: the particle flow comes from past infinity with momentum p. At time zero, the scattering happens. Then the detector will detect the particle at the future infinity, with momentum p'. It is natural to introduce two classes of Heisenberg states. One is $|p_1, \ldots, p_n\rangle_{\text{in}}$ vector denoting a state which in the far past corresponds to a collection of free state at momentum p_i. $|\vec{p_1}, \ldots, \vec{p_n}\rangle_{\text{out}}$ as above of far part replaced by far future. The scattering probability is

$$\left| {}_{\text{out}} \left\langle \vec{p_1}', \ldots, \vec{p_k}' \middle| \vec{p_1}, \ldots, \vec{p_n} \right\rangle_{\text{in}} \right|^2 . \tag{3.1}$$

If we use the abstract indices to replace the specific state and define S matrix, the matrix element is $S_{\beta\alpha}$:

$$S_{\beta\alpha} = {}_{\text{out}} \left\langle \vec{p_1}', \ldots, \vec{p_k}' \middle| \vec{p_1}, \ldots, \vec{p_n} \right\rangle_{\text{in}} = {}_{\text{out}} \langle \beta | \alpha \rangle_{\text{in}} . \tag{3.2}$$

Note we define particles as energy eigenstates of the free Hamiltonian

$$H_0 \left| \vec{p} \right\rangle = E_p \left| \vec{p} \right\rangle , \quad E_p = \sqrt{\vec{p}^2 + m^2} . \tag{3.3}$$

In and out states should be eigenstate of the full Hamiltonian $H = H_0 + V$ of the same energy, approximated by free states in far past/far future.

3.2 Defining in and out states

Match Heisenberg and Schrödinger picture states at $t = 0$, i.e. $\left|\phi^H\right\rangle = \left|\phi^S(0)\right\rangle$.

$$e^{iHt} \left|\phi^H\right\rangle = e^{iHt} \left|\phi^S(0)\right\rangle = \left|\phi^S(-t)\right\rangle . \tag{3.4}$$

Definition 3.1. *Define in and out states. We assume the following equation holds under the limit,*

$$\lim_{t\to\infty} e^{iHt} \left|\phi\right\rangle_{in} = \lim_{t\to\infty} e^{iH_0 t} \left|\phi\right\rangle , \tag{3.5}$$

likewise

$$\lim_{t\to\infty} e^{-iHt} |\phi\rangle_{out} = \lim_{t\to\infty} e^{-iH_0 t} |\phi\rangle \ . \tag{3.6}$$

With $\Omega(t) = e^{iHt}e^{-iH_0 t}$*, we have*

$$|\phi\rangle_{in} = \Omega(-\infty) |\phi\rangle \ , \tag{3.7}$$

$$|\phi\rangle_{out} = \Omega(\infty) |\phi\rangle \ . \tag{3.8}$$

In particular, introduce $|\vec{p}_1, \ldots \vec{p}_n\rangle \in \mathscr{H}_{\text{free}}$ and the in and out states has the property that

$$H |\vec{p}_1, \ldots, \vec{p}_n\rangle_{\text{in/out}} = \sum \omega_{p_i} |\vec{p}_1, \ldots, \vec{p}_n\rangle_{\text{in/out}} \ , \tag{3.9}$$

$$\lim_{t\to\pm\infty} e^{-iHt} |\vec{p_1}, \ldots, \vec{p_n}\rangle_{\text{out/in}} = \lim_{t\to\pm\infty} e^{-iH_0 t} |\vec{p}_1, \ldots, \vec{p}_n\rangle \ . \tag{3.10}$$

Remark 3.1. *To make sense of this equality,the momenta eigenstate are not localized. It needs to be interpreted as in the whole region:*

$$\lim_{t\to\pm\infty} \int f(p) e^{-iHt} |p\rangle_{out/in} dp_1 \ldots dp_n = \lim_{t\to\pm\infty} \inf f(p) e^{-iH_0 t} |\vec{p}_1, \ldots, \vec{p}_n\rangle_{in} \ . \tag{3.11}$$

Note that the equation (3.10)implies $H\Omega(\pm\infty) = \Omega(\pm\infty)H_0$ by considering the action on a basis of $\mathscr{H}_{\text{free}}$:

$$H\Omega(\pm\infty) |\vec{p}_1, \ldots, \vec{p}_n\rangle = H |\phi\rangle_{\text{out/in}} = E_\phi |\phi\rangle_{\text{out/in}} = \Omega(\pm\infty) H_0 |\vec{p}_1, \ldots, \vec{p}_n\rangle \ . \tag{3.12}$$

Rewrite the expression as

$$0 = iH\Omega(\pm\infty) - i\Omega(\pm\infty)H_0 = \frac{d}{dt}\left(e^{iHt}e^{-iH_0 t}\right)\big|_{t\to\pm\infty} \ , \tag{3.13}$$

whose interpretation is that the time evolution at early and late time is approximately free. In terms of Ω, the matrix element of S matrix is

$$S_{\beta\alpha} = {}_{\text{out}}\langle\beta|\alpha\rangle_{\text{in}} = \langle\beta| \Omega^\dagger(\infty)\Omega(-\infty) |\alpha\rangle \ . \tag{3.14}$$

It motivates us to define the operator $U(t_1, t_2)$,

Definition 3.2.

$$U(t_1, t_2) := \Omega^\dagger(t_1)\Omega(t_2) = e^{iH_0 t_1} e^{iH(t_1 - t_2)} e^{-iH_0 t_2} \, . \tag{3.15}$$

3.3 Creation and annihilation operators for in/out states

Recall that

$$|p\rangle_{\text{in/out}} = \lim_{t\to\mp\infty} \Omega(t) |p\rangle \ . \tag{3.16}$$

We want to have some operators, similar to the free case, that could create $|p\rangle_{\text{in/out}}$ acting on a vacuum. A natural Ansatz is that

$$\frac{1}{\sqrt{2\omega_p}} |p\rangle_{\text{in/out}} = a^\dagger(\mp\infty)\Omega(\mp\infty) |0\rangle \, , \tag{3.17}$$

where we define $|\Omega\rangle_{\text{in/out}}$ (up to possible normalization) as $|\Omega\rangle_{\text{in/out}} := \Omega(\mp\infty)\,|0\rangle$. Consider it carefully,

$$\Rightarrow \lim_{t\to\mp\infty} a_p^\dagger(t)\Omega(t)\,|0\rangle = \lim_{t\to\mp\infty} \Omega(t)a_{p,0}^\dagger(t)\,|0\rangle \ , \tag{3.18}$$

where the subscript 0 denotes free case. Recall that in the interacting field, the time evolution of a_p is based on the equation (1.44), then for creation operator we have

$$e^{iHt}a_p^\dagger(0)e^{-iHt} = a_p^\dagger(t)e^{ip^0t} \quad \Rightarrow \quad a_p^\dagger(t) = e^{-i\omega_p t}e^{iHt}a_p^\dagger(0)e^{-iHt} \tag{3.19}$$

So the left hand side equation becomes

$$a_p^\dagger(t)\Omega(t) = e^{-i\omega_p t}e^{iHt}a_p^\dagger(0)e^{-iHt}e^{iHt}e^{-iH_0t}. \tag{3.20}$$

Meanwhile the right hand side of the equation is

$$\Omega(t)a_{p,0}^\dagger(t) = e^{iHt}e^{-iH_0t}e^{-i\omega_p t}e^{iH_0t}a_{p,0}^\dagger(0)e^{-iH_0t}\,. \tag{3.21}$$

It is consistent to set $a_p^\dagger(0) = a_{p,0}^\dagger(0)$, which matches on interacting and free creation operator at time $t = 0$ such that

$$a_p^\dagger(t) = e^{-i\omega_p t}e^{iHt}a_{p,0}^\dagger(0)e^{-iHt} = e^{-i\omega_p t}e^{iHt}e^{-iH_0t}e^{iH_0t}a_{p,0}^\dagger(0)e^{-iH_0t}e^{iH_0t}e^{-iHt} = \Omega(t)a_{p,0}^\dagger(0)\Omega^\dagger(t)\,. \tag{3.22}$$

$$\Rightarrow \partial_t a_p^\dagger(t) = \left(\frac{d}{dt}\Omega(t)\right)a_{p,0}^\dagger(0)\Omega^\dagger(t) + \Omega(t)a_{p,0}^\dagger(0)\frac{d}{dt}\Omega^\dagger(t)\,. \tag{3.23}$$

Likewise $\lim_{t\to\mp\infty}\partial_t a_p^\dagger(t) = 0$ which is the consequence of the time evolution at early and late time is approximately free.

3.4 S-matrix, cross section and decay rate

In scattering experiment, one important physical quantity is the cross sectional area $\mathscr{A}$ of the beam. the cross section σ will be defined as

$$\frac{\sigma}{\mathscr{A}} = \left(\frac{\#\text{ of incoming particles}}{\#\text{ of scattering events}}\right)^{-1} = \text{scattering probability } \mathscr{P}\,. \tag{3.24}$$

Quantum mechanically,

$$\mathscr{P} = \frac{|\langle f|\,S\,|i\rangle\,|^2}{\langle f|i\rangle}\,. \tag{3.25}$$

If we extract the trivial scattering part $S = \mathbb{1} + iT$. Because $\langle f|j\rangle = 0$ for $f \neq j$, $\langle f|\,S\,|i\rangle = i\,\langle f|\,T\,|i\rangle$. It is often useful to extract the 4-momentum conservation δ-functor

$$T = (2\pi)^4\delta^{(4)}\left(\sum_{\text{in}} p_i - \sum_{\text{out}} p_j\right)\mathscr{M}\ , \tag{3.26}$$

$\langle f|\,\mathscr{M}\,|i\rangle$ is called the matrix element.

Example 3.1. *One important example is the $2 \to n$ scattering, the differential cross section*

$$d\sigma = \frac{1}{2E_1 2E_2 |\vec{v}_1 - \vec{v}_2|} |\mathscr{M}|^2 d\Pi\,, \tag{3.27}$$

where the $\frac{1}{2E_1 2E_2|\vec{v}_1 - \vec{v}_2|}$ is kinetic factor, $|\mathscr{M}|^2$ will be computed through Feynman diagram and $d\Pi$ is the Lorentz invariant phase factor. We define the Lorentz invariant phase space factor as

$$d\Pi_{LIPS} = (2\pi)^4 \delta^{(4)}\left(\Sigma p\right) \prod_{\textit{final states}} \frac{1}{(2\pi)^3} \frac{d^3 p_j}{2E_j}\,. \tag{3.28}$$

There's a question about the Lorentz invariance. The beam axis breaks the rotational symmetry to axial symmetry, but expected boost symmetry along this axis (usually we choose it as z-axis). Pay attention that here the $|\vec{v}_1$ -$\vec{v}_2|$ is relative velocity of the beams as view from the laboratory frame, which means that

$$2E_1 2E_2 |\vec{v}_1 - \vec{v}_2| = 4E_1 E_2 \left| \frac{p_1^z}{E_1} - \frac{p_2^z}{E_2} \right| = 4 \left| p_1^z E_2 - p_2^z E_1 \right| = 4 \left| p_1^z p_2^0 - p_2^z p_1^0 \right| = 4 |\epsilon_{\mu x y \nu} p_1^\mu p_2^\nu|\,. \tag{3.29}$$

It seems that there is some redundant indices appear. However, in this case, the only possible non-zero term is $\mu = 0, \nu = 3$ or $\mu = 3, \nu = 0$, which doesn't relate to the indices x, y, so the equation holds. The Levi-Civita antisymmetric tensor is Lorentz pseudo tensor by

$$\Lambda^\mu{}_{k_1} \Lambda^\nu{}_{k_2} \Lambda^\rho{}_{k_3} \Lambda^\sigma{}_{k_4} \epsilon_{\mu\nu\rho\sigma} = (\det \Lambda) \epsilon_{k_1 k_2 k_3 k_4}\,, \tag{3.30}$$

which proves that the kinetic factor is indeed boost invariant along the z-axis.

Example 3.2. *Another special example is the decay process, which can be viewed as the $1 \to n$ scattering process. The definition of the decay rate is*

$$\Gamma = \frac{\#\ \textit{of decays per unit time}}{\#\ \textit{of unstable particles}}\,. \tag{3.31}$$

The differential decay rate is

$$d\Gamma = \frac{1}{2E_1} |\mathscr{M}|^2 d\Pi_{LIPS}\,. \tag{3.32}$$

Chapter 4

The LSZ Reduction

LSZ is the abbreviate of three people's name: Lehmann-Symanzik-Zimmermann.

4.1 LSZ formula

S-matrix can be extracted from a time order product of quantum fields, which is exactly the LSZ formula,

$$\begin{aligned} {}_{\text{out}}\langle f|i\rangle_{\text{in}} = &\left[i\int d^4x_1 e^{-ip_1x_1}(\Box_1+m^2)\times\ldots\times i\int d^4x_m e^{+ip_mx_m}(\Box_m+m^2)\right] \\ &\times {}_{\text{out}}\langle\Omega|\,\mathscr{T}\left\{\phi(x_1)\ldots\phi(x_n)\right\}|\Omega\rangle_{\text{in}} + \text{contact terms}\,, \end{aligned} \tag{4.1}$$

where the contact terms are the contributions involving integration over $\delta^{(4)}(x_i - x_j)$ factor. The minus sign in the exponential $e^{-ip_ix_i}$ represents the incoming momentum while the plus sign represents the outgoing case. First we focus on the scalar field for simplicity. Here the $\mathscr{T}$ denotes the time ordering[1],which action of the product of field is the permute them based on the time component such that later time are to the left of earlier times.

$$\mathscr{T}\{\phi(0)\phi(|t|)\phi(-|t|)\} = \phi(|t|)\phi(0)\phi(-|t|)\,. \tag{4.2}$$

$\mathscr{T}$ just manhandles the operators within the brackets, placing them in order regardless of whether they commute or not. A question raises immediately, it this operation Lorentz invariant? We leave this question later and we will show that the spacelike separated field commute. We denote $\langle\ldots\rangle := {}_{\text{out}}\langle\Omega|\ldots|\Omega\rangle_{\text{in}}$. The factor $\Box + m^2$ becomes $-p^2 + m^2$ in Fourier space, which means that

$$\begin{aligned} \left[i\int d^4x e^{-ipx}\left(\Box+m^2\right)\ldots\right]\langle\mathscr{T}\{\ldots\}\rangle &= \left[i\int d^4x e^{-ipx}\left(-p^2+m^2\right)\ldots\right]\langle\mathscr{T}\{\ldots\}\rangle \\ &= (-p^2+m^2)i\int d^4x e^{-ipx}\ldots\langle\mathscr{T}\{\ldots\}\rangle\,, \end{aligned} \tag{4.3}$$

if the n-point function falls off sufficiently fast at ∞. These factors will therefore remove all terms in the time-ordered product except those with poles of the form $\frac{1}{p^2-m^2}$, corresponding to propagators of on-shell particles. i.e. LSZ obtains S-matrix as residues of n-point function of the Fourier transform. Now we want to prove the LSZ formula.

[1] In this note, the time ordering operator is always denoted by this notation $\mathscr{T}$.

Proof:

The proof is based on two relations,

$$i \int d^4 x e^{ipx} (\Box + m^2)\phi(x) = \sqrt{2\omega_p}\left(a_p(\infty) - a_p(-\infty)\right) , \tag{4.4}$$

$$-i \int d^4 x e^{-ipx} (\Box + m^2)\phi(x) = \sqrt{2\omega_p}\left(a_p^\dagger(\infty) - a_p^\dagger(-\infty)\right) . \tag{4.5}$$

Let us prove this lemma above first. Considering

$$i \int d^4 x e^{ipx} (\Box^2 + m^2)\phi(x) \tag{4.6}$$

we have to be careful about the boundary conditions at $t \to \pm\infty$, but we can assume that the field fall fast enough at spatial infinity, which allows us to simplify the above equation as

$$i \int d^4 x e^{ipx} (\Box^2 + m^2)\phi(x) = i \int d^4 x e^{ipx} \left(\partial_t^2 - \partial_i \partial_i + m^2\right)\phi(x) = i \int d^4 x e^{ipx} \left(\partial_t^2 + \vec{p}^2 + m^2\right)\phi(x) \,, \tag{4.7}$$

since

$$\begin{aligned} i \int d^4 x e^{ipx} (-\partial_i \partial_i)\phi(x) &= -\, \phi(x)\partial_i e^{ipx}\big|_{-\infty}^{+\infty} + \int d^4 x (ip_i) e^{ipx} \partial_i \phi(x) \\ &= -(ip_i)^2 \int d^4 x e^{ipx} \phi(x) = \int d^4 x \vec{p}^2 e^{ipx} \phi(x) \,. \end{aligned} \tag{4.8}$$

Using the notation $\omega_p^2 = \vec{p}^2 + m^2$, the integral is

$$i \int_{-\infty}^{+\infty} dt e^{i\omega_p t} \int d^3 x e^{-i\vec{p}\cdot\vec{x}} \left(\partial_t^2 + \omega_p^2\right)\phi(x) = \int_{-\infty}^{+\infty} dt \partial_t \left[e^{i\omega_p t} \int d^3 x e^{-i\vec{p}\cdot\vec{x}} \left(i\partial_t + \omega_p\right)\phi(x) \right] , \tag{4.9}$$

as

$$\partial_t \left[e^{i\omega_p t} (i\partial_t + \omega_p)\phi(x) \right] = e^{i\omega_p t} \left(i\omega_p \left(\cancel{i\partial_t} + \omega_p \right) + i\partial_t^2 + \cancel{\omega_p \partial_t} \right) \phi(x) \,. \tag{4.10}$$

The red part of the equation (4.9) is

$$\int d^3 x e^{-i\vec{p}\cdot\vec{x}} \left(i\partial_t + \omega_p\right)\phi(x) = \int d^3 x e^{-i\vec{p}\cdot\vec{x}} (i\partial_t + \omega_p) \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left[a_k(t) e^{-ikx} + a_k^\dagger(t) e^{ikx} \right] . \tag{4.11}$$

We want to evaluate this expression when $t \to \mp\infty$, recall that $\lim_{t\to\mp\infty} \partial_t a_p^\dagger(t) = 0, \lim_{t\to\mp\infty} \partial_t a_p(t) = 0$, after the action of the time derivative,

$$\Rightarrow \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left[(\omega_k + \omega_p) a_k(t) e^{-i\omega_k t} \int d^3 x e^{i(\vec{k}-\vec{p})\cdot\vec{x}} + (-\omega_k + \omega_p) a_k^\dagger e^{i\omega_k t} \int d^3 x e^{-i(\vec{k}+\vec{p})\cdot\vec{x}} \right] , \tag{4.12}$$

The integration of the spatial indices is easy, which generates two δ function,

$$\Rightarrow \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left[(\omega_k + \omega_p) a_k(t) e^{-i\omega_k t} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) + (-\omega_k + \omega_p) a_k^\dagger e^{i\omega_k t} (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{p}) \right] . \tag{4.13}$$

Completing the integration and notice that $\omega_{\vec{p}} = \omega_{-\vec{p}}$, the $a_k^\dagger$ part vanishes. The left term is

$$\Rightarrow \sqrt{2\omega_p} a_p(t) e^{-i\omega_p t} \,. \tag{4.14}$$

Substitute it into the equation (4.9),

$$\Rightarrow i \int d^4x e^{ipx} (\Box^2 + m^2)\phi(x) = \int_{-\infty}^{+\infty} dt \partial_t \left[e^{i\omega_p t} \sqrt{2\omega_p} a_p(t) e^{-i\omega_p t} \right] = \sqrt{2\omega_p} \left(a_p(\infty) - a_p(-\infty) \right) \,. \tag{4.15}$$

Likewise

$$i \int d^4x e^{-ipx} (\Box + m^2)\phi(x) = -\sqrt{2\omega_p} \left(a_p^\dagger(\infty) - a_p^\dagger(-\infty) \right) \,. \tag{4.16}$$

Back to LSZ, consider the case of $2 \to n-2$ scattering,

$$|i\rangle_{\text{in}} = \sqrt{2\omega_1}\sqrt{2\omega_2} a_{p_1}^\dagger(-\infty)\, a_{p_2}^\dagger(-\infty)\, |\Omega\rangle_{\text{in}} \,, \tag{4.17}$$

$$|f\rangle_{\text{out}} = \prod_{k=3}^{n} \sqrt{2\omega_k} a_{p_k}^\dagger(\infty)\, |\Omega\rangle_{\text{out}} \,. \tag{4.18}$$

The S matrix element

$$\begin{aligned} {}_{\text{out}}\langle f|i\rangle_{\text{in}} &= \left(\prod_{k=1}^{n} \sqrt{2\omega_k} \right) {}_{\text{out}}\langle\Omega|\, a_{p_3}(\infty) \dots a_{p_n}(\infty) a_{p_1}^\dagger(-\infty) a_{p_2}^\dagger\, |\Omega\rangle_{\text{in}} \\ &= \left(\prod_{k=1}^{n} \sqrt{2\omega_k} \right) {}_{\text{out}}\langle\Omega|\, \mathscr{T}\{ a_{p_3}(\infty) \dots a_{p_n}(\infty) a_{p_1}^\dagger(-\infty) a_{p_2}^\dagger \}\, |\Omega\rangle_{\text{in}} \,, \end{aligned} \tag{4.19}$$

where the second equal sign holds because it is automatically time ordered. The next step is to make replacement: for arbitrary $a_{p_i}(\infty)$, we replace it by $a_{p_i}(\infty) - a_{p_i}(-\infty)$. For arbitrary $a_{p_j}^\dagger(-\infty)$, we replace it by $a_{p_j}^\dagger(-\infty) - a_{p_j}^\dagger(\infty)$. The replacement takes the advantage of the time ordering product, since the $a_{p_i}(-\infty)$ will move to the right and annihilate the state and $a_{p_j}^\dagger(+\infty)$ will move to the left and annihilate the state, too. We will exclude trivial scattering, i.e. $p_1, p_2 \in \{p_3, \dots, p_n\}$. Plugging in the operator expansions for $a_{p_3}(\infty) - a_{p_3}(-\infty)$, ... , $a_{p_1}^\dagger(-\infty) - a_{p_1}^\dagger(\infty)$ yields LSZ up to δ-function contributions arising upon pulling ∂_t^2 out of the $\mathscr{T}$-product.

Chapter 5

Computing time ordered products

5.1 The Feynman propagator

In this section we will evaluate the free 2-point function. The strategy is to move the annihilation operator a next to the creation operator $a^\dagger$.

5.1.1 Free scalar field

In this section, we ignore the subscript 0 for simplicity, but the reader should keep in mind that we are discussing the free scalar field. The free scalar field is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{-ikx} + a_k^\dagger e^{ikx} \right), \, k^0 = \omega_k = \sqrt{\vec{k}^2 + m^2} \, . \tag{5.1}$$

We want to compute the 2 point function $\langle 0 | \mathscr{T} \{\phi(x_1)\phi(x_2)\}$,

$$\begin{aligned} \langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle &= \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_1}}} \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_2}}} \langle 0 | a_{k_1} a_{k_2}^\dagger | 0 \rangle \, e^{-ik_1x_1 + ik_2x_2} \\ &= \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_1}}} \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_2}}} (2\pi)^3 \delta^{(3)}(\vec{k_1} - \vec{k_2}) e^{-ik_1x_1 + ik_2x_2} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-ik(x_1 - x_2)} \, . \end{aligned} \tag{5.2}$$

Next, we introduce time ordering,

$$\begin{aligned} \langle 0 | \mathscr{T} \{\phi(x_1)\phi(x_2)\} | 0 \rangle &= \langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle \, \Theta(t_1 - t_2) + \langle 0 | \phi(x_2)\phi(x_1) | 0 \rangle \, \theta(t_2 - t_1) \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} e^{-i\omega_k(t_1 - t_2)} \Theta(t_1 - t_2) + e^{-i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} e^{i\omega_k(t_1 - t_2)} \Theta(t_2 - t_1) \right] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} \left[e^{-i\omega_k\tau} \Theta(\tau) + e^{i\omega\tau} \Theta(-\tau) \right] \, , \end{aligned} \tag{5.3}$$

where we have defined $t_1 - t_2 = \tau$. We need to check that if it is Lorentz invariant. First, we claim that

$$e^{-i\omega_k\tau}\Theta(\tau) + e^{i\omega_k\tau}\Theta(-\tau) = \lim_{\epsilon \to 0} \left(-\frac{2\omega_k}{2\pi i} \right) \int_{-\infty}^{+\infty} \frac{d\omega}{\omega^2 - \omega_k^2 + i\epsilon} e^{i\omega\tau} \, . \tag{5.4}$$

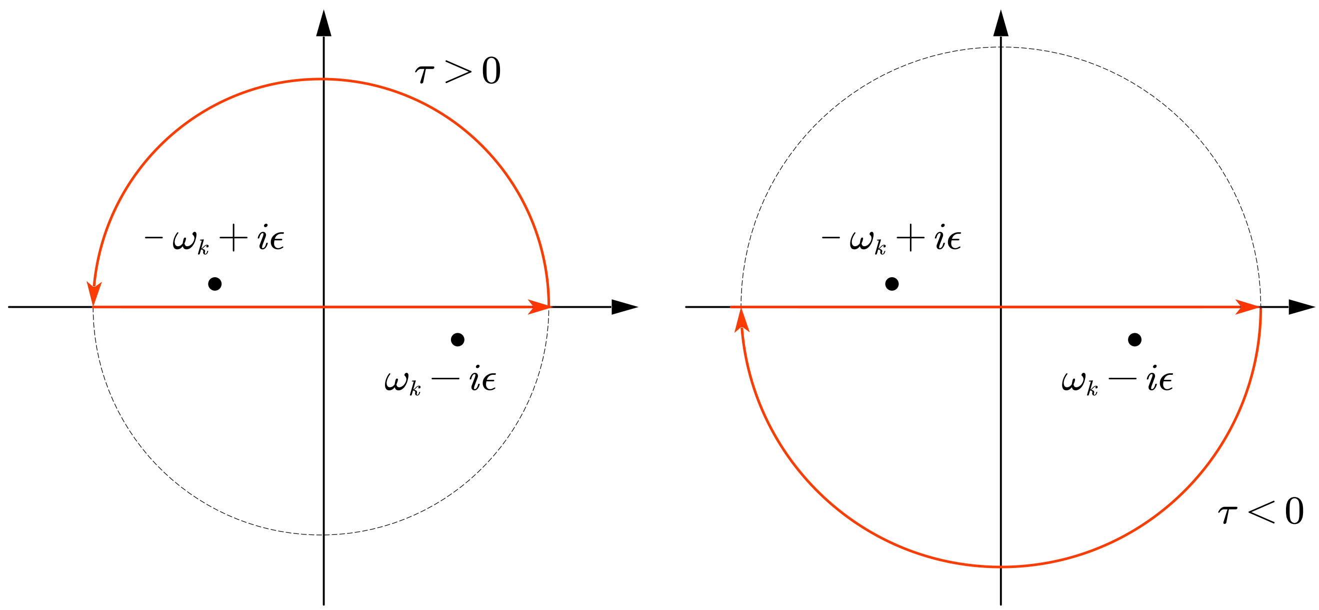


Figure 5.1: The integral contour

Proof

: We will prove this expression by evaluating via residue theorem. We need to specify poles of the function,

$$\begin{aligned} \frac{1}{\omega^2 - \omega_k^2 + i\epsilon} &= \frac{1}{\omega - (\omega_k - i\epsilon')} \times \frac{1}{\omega + (\omega_k - i\epsilon')} \quad \text{with} \quad 2\omega_k \epsilon' = \epsilon \\ &= \frac{1}{2\omega_k} \left[\frac{1}{\omega - (\omega_k - i\epsilon')} - \frac{1}{\omega - (-\omega_k + i\epsilon')} \right] . \end{aligned} \tag{5.5}$$

Consider two cases

- $\tau > 0$, the integral will be

$$I = \int_{\tau > 0} = 2\pi i \text{Res} \frac{-1}{\omega_k} \frac{e^{i\omega\tau}}{\omega - (-\omega_k + i\epsilon)} = 2\pi i \frac{-1}{2\omega_k} e^{-i\omega_k \tau} . \tag{5.6}$$

- $\tau < 0$, the integral is

$$I = \int_{\tau < 0} = 2\pi i \text{Res} \frac{1}{2\omega_k} \frac{e^{i\omega\tau}}{\omega - (\omega_k - i\epsilon')} = 2\pi i \frac{1}{\omega_k} e^{i\omega_k \tau} . \tag{5.7}$$

Based on these two calculations, we can see that

$$\Rightarrow \quad \left(-\frac{2\omega_k}{2\pi i} \right) \int_{-\infty}^{+\infty} \frac{d\omega}{\omega^2 - \omega_k^2 + i\epsilon} e^{i\omega\tau} = e^{-i\omega_k \tau} \Theta(\tau) + e^{i\omega_k \tau} \Theta(-\tau) , \tag{5.8}$$

analytically extend the equation we see that the limit also satisfies.

$$\begin{aligned}\Rightarrow \langle 0| \mathscr{T}\left\{\phi(x_1)\phi(x_2)\right\}|0\rangle &= \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\left(-\frac{2\omega_k}{2\pi i}\right)\int_{-\infty}^{+\infty}\frac{d\omega}{\omega^2-\omega_k^2+i\epsilon}e^{i\omega\tau} \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{(k^0)^2-(\vec{k}^2+m^2)+i\epsilon}e^{ik^0(t_1-t_2)-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik(x_1-x_2)}\,,\end{aligned} \tag{5.9}$$

which is manifestly Lorentz invariant. Terminally time ordered vacuum expectation value (VEV) of two free fields (i.e. their free 2 point function) is called the Feynman propagator $D_F(x_1-x_2)$. Note that $D_F(x_1-x_2)$ above is a Green's function to the EOM of the free field $(\Box+m^2)D_F(x_1-x_2) = -i\delta^{(4)}(x-y)$. We will see that this holds generally later. The $i\epsilon$ description fixes the choice of adding a homogeneous solution to the Green's function.

5.2 Calculating n-point function in interacting theories

Recall that in Heisenberg picture,

$$i\partial_t\mathscr{O}(x) = [\mathscr{O}(x), H] \quad \Rightarrow \quad \mathscr{O}(x) = e^{iHt}\mathscr{O}(0,\vec{x})e^{-iHt}. \tag{5.10}$$

If we consider the perturbation theory, $H = H_0(t) + V(t)$, where the interacting part $V(t)$ is small. Here we use the Heisenberg picture

$$H_0(t) = e^{iHt}H_0(0)e^{-iHt}\,, V(t) = e^{iHt}V(0)e^{-iHt}\,. \tag{5.11}$$

To relate the free theory, we introduce the interaction pictures. The Heisenberg field is $\phi(x) = e^{iHt}\phi(0,\vec{x})e^{-iHt}$, the interacting picture is

$$\phi_I(x) = e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t} = \phi_0(x)\,, \tag{5.12}$$

where the $\phi_0(x)$ represents the free field. If we transform it back to the Schrödinger picture

$$\phi(t,\vec{x}) = e^{iHt}e^{-iH_0t}\phi_I(x)e^{iH_0t}e^{-iHt} = \Omega(t)\phi_0(x)\Omega^\dagger(t)\,. \tag{5.13}$$

Recall that

$$\begin{aligned}|\Omega\rangle_{\text{in}} &= \mathscr{N}_I\Omega(-\infty)|0\rangle\ , \\ |\Omega\rangle_{\text{out}} &= \mathscr{N}_F\Omega(\infty)|0\rangle\ ,\end{aligned} \tag{5.14}$$

where $\mathscr{N}_I$ and $\mathscr{N}_F$ are normalization factor, which satisfies $1 = {}_{\text{out}}\langle\Omega|\Omega\rangle_{\text{in}} = \mathscr{N}_I\mathscr{N}_F^* = \langle 0|\Omega^\dagger(\infty)\Omega(-\infty)|0\rangle$. With the expression above, we can write the vacuum expectation of time order product as

$$\begin{aligned}&{}_{\text{out}}\langle\Omega|\mathscr{T}\left\{\phi(x_1)\ldots\phi(x_n)\right\}|\Omega\rangle_{\text{in}} = \\ &\mathscr{N}_I\mathscr{N}_F^*\langle 0|\Omega^\dagger(\infty)\Omega(t_1)\phi_0(x_1)\Omega^\dagger(t_1)\Omega(t_2)\phi_0(x_2)\Omega^\dagger(t_2)\ldots\Omega(t_n)\phi_0(x_n)\Omega^\dagger(t_n)\Omega(-\infty)|0\rangle\ ,\end{aligned} \tag{5.15}$$

where we have assume that $t_1 > t_2 > \ldots > t_n$ without loss of generality. Define

$$U(t_1, t_2) = \Omega^\dagger(t_1)\Omega(t_2) = e^{iH_0 t_1} e^{-iH(t_1-t_2)} e^{-iH_0 t_2}\,, \tag{5.16}$$

and U satisfies the following differential equation

$$\begin{aligned} i\partial_t U(t, t_0) &= i\left(iH_0 e^{iH_0 t} e^{-iH(t-t_0)} e^{-iHt_0} + e^{iH_0 t}(-iH) e^{-iH(t-t_0)} e^{-iH_0 t_0}\right) \\ &= e^{iH_0 t} V e^{-iH_0 t} U(t, t_0) \\ &= V_I(t) U(t, t_0)\,. \end{aligned} \tag{5.17}$$

We could solve this equation iteratively,

$$U(t, t_0) = \mathbb{1} - i\int_{t_0}^{t} dt' V_I(t') U(t', t_0)\,. \tag{5.18}$$

If V is small, it is safe to expand the integral

$$\begin{aligned} U(t, t_0) &= \mathbb{1} - i\int_{t_0}^{t} dt' V_I(t') \left(\mathbb{1} - i\int_{t_0}^{t'} dt'' V_I(t'') (\ldots)\right) \\ &= \mathbb{1} - i\int_{t_0}^{t} dt' V_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V_I(t') V_I(t'') + \ldots\,. \end{aligned} \tag{5.19}$$

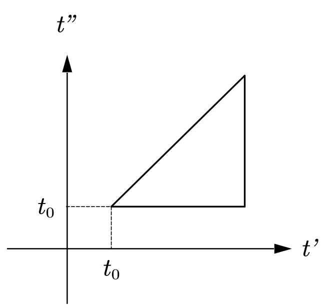


Figure 5.2: Integration region

Note that we have to impose $t' \geq t''$. The integration region of the second term is as the figure (5.2) shows. It will simplify our considerations to integrate over the full space. This is justified if we

1. divided by the factor 2.
2. reverse the orders of the operator when $t'' > t'$, i.e. if we time-order the integral.

Then we have the following equation

$$U(t, t_0) = \mathbb{1} - i\int_{t_0}^{t} dt' V_I(t') + \frac{1}{2}(-i)^2 \int_{t_0}^{t}\int_{t_0}^{t} dt'' \mathscr{T}\left\{V_I(t') V_I(t'')\right\} + \ldots \tag{5.20}$$

By reduction we have

$$\begin{aligned} U(t, t_0) &= \mathbb{1} - i\int_{t_0}^{t} dt' V_I(t') + \ldots + \frac{(-i)^n}{n!}\int_{t_0}^{t} \ldots \int_{t_0}^{t} dt_1 \ldots dt_n \mathscr{T}\left\{V_I(t_1) \ldots V_I(t_n)\right\} + \ldots \\ &= \mathscr{T}\left\{\exp\left[-i\int_{t_0}^{t} dt' V_I(t')\right]\right\}\,. \end{aligned} \tag{5.21}$$

This is known as a time-ordered exponential or a Dyson series.

Remark 5.1. *Based on the definition of $U(t_1, t_2)$, we have the following relation*

$$U(t_1,t_2)U(t_2,t_3) \overset{t_1>t_2>t_3}{=} e^{iH_0t_1}e^{-iH(t_1-t_2)}e^{-iH_0t_2}e^{iH_0t_2}e^{-iH(t_2-t_3)}e^{-iH_0t_3} = U(t_1,t_3)\,. \tag{5.22}$$

If we use Dyson series to prove that, it is obvious taking the advantage of exponential function.

Returning our calculation of the VEV or the time order product of the interacting fields. We have

$$\begin{aligned}
&{}_{\text{out}}\langle\Omega|\,\mathscr{T}\left\{\phi(x_1)\dots\phi(x_n)\right\}|\Omega\rangle_{\text{in}} = \\
&\mathscr{N}_I\mathscr{N}_F^*\,\langle 0|\,\Omega^\dagger(\infty)\Omega(t_1)\phi_0(x_1)\Omega^\dagger(t_1)\Omega(t_2)\phi_0(x_2)\Omega^\dagger(t_2)\dots\Omega(t_n)\phi_0(x_n)\Omega^\dagger(t_n)\Omega(-\infty)\,|0\rangle \\
&= \mathscr{N}_I\mathscr{N}_F^*\,\langle 0|\,\mathscr{T}\left\{U(\infty,t_1)\phi_I(x_1)U(t_1,t_2)\dots\phi_I(x_n)U(t_n,-\infty)\right\}|0\rangle\ .
\end{aligned} \tag{5.23}$$

Thanks to the property of time order product $\mathscr{T}$ that no matter what order of the operators, the final result will be listed based on the time component. So the operators commute in the time order product, we rearrange the equation as

$$\begin{aligned}
&\mathscr{N}_I\mathscr{N}_F^*\,\langle 0|\,\mathscr{T}\left\{\phi_I(x_1)\dots\phi_I(x_n)U(\infty,t_1)U(t_1,t_2)\dots U(t_n,\infty)\right\}|0\rangle = \\
&\qquad \mathscr{N}_I\mathscr{N}_F^*\,\langle 0|\,\mathscr{T}\left\{\phi_I(x_1)\dots\phi_I(x_n)\mathscr{T}\left\{\exp\left[-i\int_{-\infty}^{+\infty}V_I(t)dt\right]\right\}\right\}|0\rangle\ ,
\end{aligned} \tag{5.24}$$

where we have already taken the advantage of the property Eq. (5.22). In time order product, the time order product operations can be removed since the result will be the same. We fix the normalization factor as

$$\mathscr{N}_I\mathscr{N}_F^* = \frac{1}{\langle 0|\,\mathscr{T}\left\{\exp\left[-i\int_{-\infty}^{+\infty}V_I(t)dt\right]\right\}|0\rangle}\,. \tag{5.25}$$

We finally arrive at

$${}_{\text{out}}\langle\Omega|\,\mathscr{T}\left\{\phi(x_1)\dots\phi(x_n)\right\}|\Omega\rangle_{\text{in}} = \frac{\langle 0|\,\mathscr{T}\left\{\phi_I(x_1)\dots\phi_I(x_n)\exp\left[-i\int_{-\infty}^{+\infty}V_I(t)dt\right]\right\}|0\rangle}{\langle 0|\,\mathscr{T}\left\{\exp\left[-i\int_{-\infty}^{+\infty}V_I(t)dt\right]\right\}|0\rangle}\,. \tag{5.26}$$

For a physics theory, if we know the Lagrangian $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}$, the potential part will be

$$V = -\int d^3x\mathscr{L}_{\text{int}}(\phi)\ \Rightarrow\ V_I = e^{iH_0t}Ve^{-iH_0t} = -\int d^3x\mathscr{L}_{\text{int}}(\phi_I)\,.\,. \tag{5.27}$$

Combine with the time integration, the exponential factor will be

$$\exp\left[-i\int_{-\infty}^{+\infty}V_I(t)dt\right] = \exp\left[i\int d^4x\mathscr{L}_{\text{int}}\right]\,. \tag{5.28}$$

Chapter 6

Hamiltonian derivation of Feynman rules

6.1 General strategy

We will take an example to illustrate the calculation method constructed above. Consider the Lagrangian of ϕ^3 theory

$$\mathscr{L}_{\text{int}} = \frac{g}{3!}\phi^3\,. \tag{6.1}$$

The interacting 2-point function is

$$\begin{aligned}
{}_{\text{out}}\langle\Omega|\,\mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\right\}|\Omega\rangle_{\text{in}} &= \langle 0|\,\mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\exp\left(i\int_{-\infty}^{+\infty} d^4x\mathscr{L}_{\text{int}}(\phi_I)\right)\right\}|0\rangle \\
&= \langle 0|\,\mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\right\}|0\rangle + \frac{g}{3!}\int_{-\infty}^{+\infty} d^4x\,\langle 0|\,\mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\phi_I(x)^3\right\}|0\rangle + \ldots\,.
\end{aligned} \tag{6.2}$$

we need to evaluate the free n-point function in the interacting picture, with n increasing with the order in the coupling constant g. We evaluate it at the level of creation and annihilation operator. We write $\phi_I(x) = \phi_+(x) + \phi_-(x)$ with

$$\phi_+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}e^{ipx}a_p^\dagger\,,\,\phi_-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}e^{-ipx}a_p\,. \tag{6.3}$$

The strategy is to move all the ϕ_- operators past all the ϕ_+ operators since ϕ_- will annihilate the vacuum.

Definition 6.1. *Normal ordering: A product of creation and annihilation operators is called normal ordered, if all annihilation operators are to the right of creation operators. The normal ordering operations is denoted as* $:\ldots:$*, defined as*

$$:(a_p^\dagger + a_p)(a_k^\dagger + a_k):= a_p^\dagger a_k^\dagger + a_p^\dagger a_k + :a_p a_k^\dagger: + a_p a_k = a_p^\dagger a_k^\dagger + a_p^\dagger a_k + a_k^\dagger a_p + a_p a_k\,. \tag{6.4}$$

The fundamental property of $:\ldots:$ is that the VEV of non-trivial normal ordered product vanishes.

Definition 6.2. *Contracting two fields $\phi_I(x_i)$, $\phi_I(x_j)$ in a product of fields means removing them from the product, and introducing a factor of the associated Feynman propagator $D_F(x_i - x_j)$ instead.*

6.2 Wick theorem

Theorem 6.1. *Wick theorem: The time order product*

$$\mathscr{T}\left\{\phi_I(x_1)\ldots\phi_I(x_n)\right\} =: \phi_I(x_1)\ldots\phi_I(x_n) : + : \textit{all possible contraction} : , \tag{6.5}$$

where $t_i \neq t_j$ *for* $i \neq j$.

Example 6.1.

$$\begin{aligned}
&\mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\right\} =: \phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4) : \\
&+ D_F(x_1-x_2) : \phi_I(x_3)\phi_I(x_4) : + D_F(x_1-x_3) : \phi_I(x_2)\phi_I(x_4) : \\
&+ D_F(x_1-x_4) : \phi_I(x_2)\phi_I(x_3) : + \ldots (\textit{total 6 similar terms}) \\
&+ D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3) + D_F(x_1-x_4)D_F(x_2-x_3)\,.
\end{aligned} \tag{6.6}$$

Proof:

We prove this theorem by reduction. First we prove it is true when $n=2$.

$$\begin{aligned}
\mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\right\} &= \phi_I(x_1)\phi_I(x_2)\Theta(t_1-t_2) + \phi_I(x_2)\phi_I(x_1)\Theta(t_2-t_1) \\
&= (: \phi_I(x_1)\phi_I(x_2) : + [\phi_-(x_1), \phi_+(x_2)])\,\Theta(t_1-t_2) \\
&+ (: \phi_I(x_2)\phi_I(x_1) : + [\phi_-(x_2), \phi_+(x_1)])\Theta(t_2-t_1) \\
&=: \phi_I(x_1)\phi_I(x_2) : + [\phi_-(x_1), \phi_+(x_2)]\Theta(t_1-t_2) + [\phi_-(x_2), \phi_+(x_1)]\Theta(t_2-t_1)\,.
\end{aligned} \tag{6.7}$$

The last two terms are exactly the Feynman propagator, as

$$\begin{aligned}
&D_F(x_1-x_2) = \langle 0|\,\mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\right\}|0\rangle \\
&= \langle 0|\,(\phi_+(x_1)+\phi_-(x_1))(\phi_+(x_2)+\phi_-(x_2))\Theta(t_1-t_2) + (\phi_+(x_2)+\phi_-(x_2))(\phi_+(x_1)+\phi_-(x_1))\Theta(t_1-t_2)\,|0\rangle \\
&= \langle 0|\,[\phi_-(x_1), \phi_+(x_2)]\Theta(t_1-t_2) + [\phi_-(x_2), \phi_+(x_1)]\Theta(t_2-t_1)\,|0\rangle\ .
\end{aligned} \tag{6.8}$$

Because the commutator $[a_p, a_q^\dagger] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$ is no longer an operator (or proportional to identity operator) anymore, the commutator $[\phi_-(x_1), \phi_+(x_2)]$ is proportional to the identity operator, which means that

$$D_F(x_1-x_2) = [\phi_-(x_1), \phi_+(x_2)]\Theta(t_1-t_2) + [\phi_-(x_2), \phi_+(x_1)]\Theta(t_2-t_1)\,. \tag{6.9}$$

In this way, we prove that the Wick's theorem is true when $n=2$,

$$\mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\right\} =: \phi_I(x_1)\phi_I(x_2) : + D_F(x_1-x_2)\,. \tag{6.10}$$

Now we assume that Wick's theorem holds for $n-1$ fields, and let $t_1 > t_i$ for $i = 2, \ldots, n$. Then

$$\begin{aligned}
\mathscr{T}\left\{\phi_I(x_1)\ldots\phi_I(x_n)\right\} &= \phi_I(x_1)\mathscr{T}\left\{\phi_I(x_2)\ldots\phi_I(x_n)\right\} \\
&= (\phi_+(x_1)+\phi_-(x_1))) : \phi_I(x_2)\ldots\phi_I(x_n) : + \text{all possible contractions}\,,
\end{aligned} \tag{6.11}$$

where we use the assumption that $t_1 > t_i$ and the Wick theorem for $n-1$ fields. Notice that for creation operator, it is safe to move it into the normal order,

$$\phi_+(x_1) : \phi_I(x_{i_1}) \dots \phi_I(x_{i_k}) :=: \phi_+(x_1)\phi_I(x_{i_1}) \dots \phi_I(x_{i_k}) : . \tag{6.12}$$

For the annihilation operator

$$\begin{aligned} &\phi_-(x_1) :\phi_I(x_{i_1}) \dots \phi_I(x_{i_k}) := \\ &\qquad : \phi_-(x_1)\phi_I(x_{i_1}) \dots \phi_I(x_{i_k}) : + \text{k contractions } \phi_-(x_1) \text{ with } \phi_I(x_{i_j}) \quad j = 1, \dots k\,. \end{aligned} \tag{6.13}$$

Every exchange of $\phi_-(x_1)$ and $\phi_+(x_{i_j})$ will generate a commutator $[\phi_-(x_1), \phi_I(x_{i_j})]$. The non trivial commutator is

$$[\phi_-(x_1), \phi_+(x_{i_j})] \overset{t_1 > t_{i_j}}{=} [\phi_-(x_1), \phi_+(x_{i_j})]\Theta(t_1 - t_{i_j}) = D_F(x_1 - x_{i_j})\,, \tag{6.14}$$

which generates the right hand sides of the Wick's theorem for n fields. As

$$\langle 0| : \prod_{i=1}^{k} \phi_I(x_i) : |0\rangle = 0\,, \tag{6.15}$$

for $k > 0$, only fully contracted terms on the right hand side of Wick's theorem contribute to VEV. We finish the proof.

Remark 6.1. *When considering the time order product $\mathscr{T}\{\phi_I(x_1) \dots \phi_I(x_n)\mathscr{O}(x_{n+1})\}$, with $\mathscr{O}(x_{n+1})$ a polynomial in $\phi_I(x_{n+1})$, we need to give a make-sense definition of the time order product of $\mathscr{T}\{\phi_I(x_1) \dots \phi_I(x_n)\phi_I(x_{n+1}) \dots \phi_I(x_{n+1})\}$, there are two options to consider this problem*

- *Consider normal operators $\mathscr{O}(x_{n+1}) =: \phi_I(x_{n+1})^n :$*
- *Interpret as*
$$\phi_I(x)^n = \lim_{\epsilon \to \pm 0} \phi_I(t + (n-1)\epsilon, \vec{x}) \dots \phi_I(t, \vec{x})\,. \tag{6.16}$$

6.3 General lessons from ϕ^3-theory

Example 6.2. *Two-point function in ϕ^3-theory with $\mathscr{L}_{int} = \frac{g}{3!}\phi^3$, perturbatively expanding the VEV of time order product and list the terms in order of g*

- g^0: $\langle 0| \mathscr{T}\{\phi_I(x_1)\phi_I(x_2)\} |0\rangle = D_F(x_1 - x_2)$
- g^1: $\frac{i}{3!}\int d^4x \langle 0| \mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\phi_I(x)^3\right\} |0\rangle = 0$ *as no possible complete contractions.*
- g^2: *the first non-trivial contribution at this point*
$$\frac{(i)^2}{2!}\left(\frac{1}{3!}\right)\int d^4x \int d^4y \langle 0| \mathscr{T}\left\{\phi_I(x_1)\phi_I(x_2)\phi_I(x)^3\phi_I(y)^3\right\} |0\rangle\,, \tag{6.17}$$

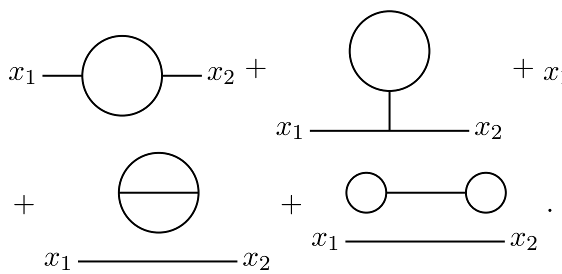
  we need to count all the possible contractions term and we will introduce Feynman diagram to keep track of it.

6.3.1 Feynman diagrams

Here are the basic building blocks of Feynman diagrams

1. one vertex per internal coordinate $(x, y, \ldots) \to$ the number of the vertices equal to the order in g. For example, if we have two internal point, the contribution of this diagram is of order g^2.
2. one point per external field, labelled by the spacetime coordinate $(x_1, \ldots, x_n)$.
3. Contractions are indicated by lines connecting the corresponding points/vertices.
4. The valency of vertices (number of lines connected to it): the number of fields in corresponding monomial in $\mathscr{L}_{\text{int}}$.

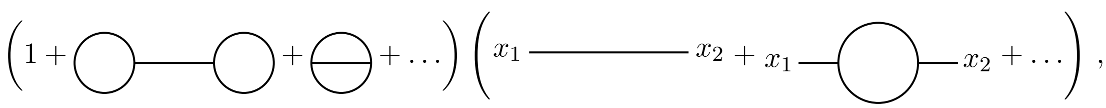
Example 6.3. *In ϕ^3-theory, at the order of g^2, the Feynman diagrams are*



(6.18)

6.3.2 Bubbles

Bubbles are subgraphs that do not involve external points $(x_1, \ldots, x_n)$, and they can be factored out like



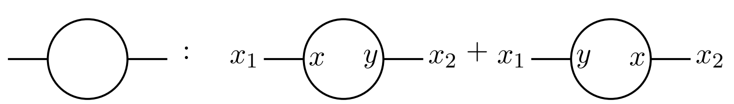
(6.19)

where the first factor will be cancelled against the denominator.

6.3.3 Prefactor

- The $\frac{1}{n!}$ from expanding $e^{i \int_{-\infty}^{+\infty} d^4x \mathscr{L}_{\text{int}}}$ generically cancels against $n!$ permutations of internal coordinates giving same contributions. The factor n corresponds to the order of the coupling constant, which corresponds to the number of the internal points. The permutation of the internal point then cancels the factor from Taylor expansion.

  Example 6.4. *For example the Feynman diagram below actually represents two diagram*



(6.20)

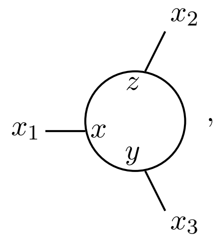
- Additional multiplicities: due to identical fields at vertex. It is conventional to normalize interaction terms by dividing by the number of the identical fields associate to a vertex.

  Example 6.5. *For example, the interaction terms are*

$$\frac{g}{3!}\phi^3\,, \quad \frac{\lambda}{4!}\phi^4\,, \quad \frac{\mu}{2!3!5!}\phi_1^2\phi_2^3\phi_3^5\,. \tag{6.21}$$

Generically, these prefactors will cancel against multiplicities.

Example 6.6. *The Feynman diagram as the equation shows*

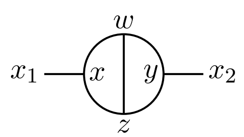


, (6.22)

comes from the term $\phi(x_1)\phi(x_2)\phi(x_3)\phi(x)^3\phi(y)^3\phi(z)^3$*, there are* $(3!)^3$ *contraction corresponding to this contraction pattern, which cancels the factor* $\left(\frac{1}{3!}\right)^3$*.*

However, when the Feynman diagram exhibit symmetries, these cancellations can be incomplete. We need the symmetry factors.

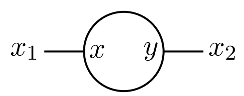
Example 6.7. *The example of* $\frac{1}{n!}$ *from* $e^{i\int d^4x\mathscr{L}_{int}}$ *not cancelling*



, (6.23)

the contraction of point x_1 *with the internal point has 4 choice, the* x_2 *has 3 choice, which* $4\times 3/4! = 1/2$*. The symmetry factor is* $\frac{1}{2}$*.*

Example 6.8. *Example of* $\frac{1}{n!}$ *from normalization of interaction coupling constant not cancelling,*

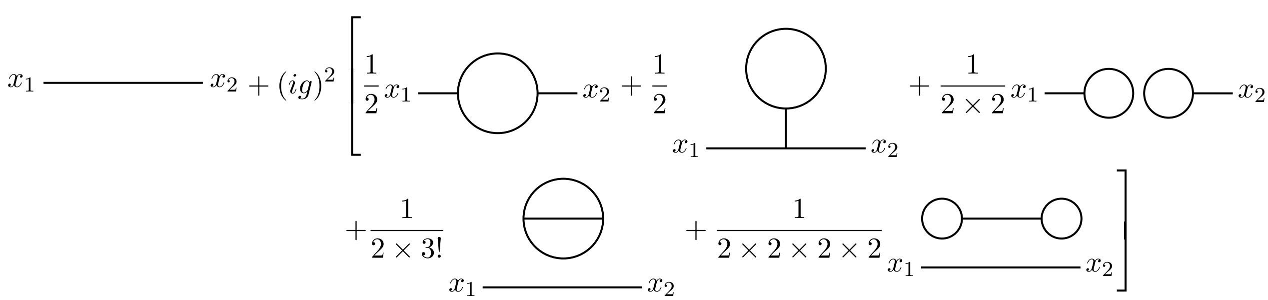


. (6.24)

The contraction is

$$\phi(x_1)\phi(x_2)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\,,\qquad \left(\frac{1}{3!}\right)^2(3!3) = \frac{1}{2}\,. \tag{6.25}$$

Back to the 2-point function in ϕ^3, up the order g^2 has tree level Feynman diagram is



(6.26)

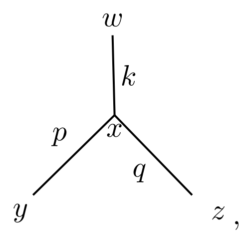
6.3.4 Momentum space Feynman rules for time ordered products

A vertex labelled by the spacetime variable x **comes with**

- a factor of ig .

- a $\int d^4x$ integration over the attached propagators.

Example 6.9. *The Feynman diagram below,*



(6.27)

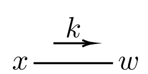
if we do the Fourier transformation, we have

$$
\begin{aligned}
&ig\int d^4x D_F(x-w)D_F(x-y)D_F(x-z) = ig\times \\
&\int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik(x-w)}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{ip(x-y)}\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}e^{iq(x-z)}\,.
\end{aligned}
\tag{6.28}
$$

The integral of x can be performed, by using the identity

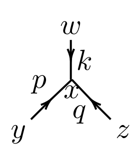
$$
\int d^4x e^{ix(k+p+q)} = (2\pi)^4\delta^{(4)}(k+q+p)\,. \tag{6.29}
$$

This tell us that the spacetime integral at vertex gives rise to momentum δ function. To keep trace of signs, we will call e^{-ikx} incoming momentum at the vertex labelled by x, e^{ikx} outgoing momentum at vertex x. We hence accompany lines corresponding to propagators by arrows,



(6.30)

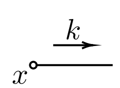
where k is outgoing at x, incoming at w. We then have



$$
= (2\pi)^4\delta^{(4)}(k+p+q)\,, \tag{6.31}
$$

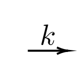
if the direction is inverse, the sign is also inverse. This tells us momentum is conserved at each vertex.

An external spacetime is not integrated. Each such point comes with a factor of e^{ikx} or e^{-ikx} from the attached propagator.



(6.32)

All connected lines come with factors



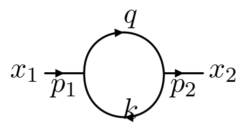
$$
= \frac{i}{k^2-m^2+i\epsilon}\,. \tag{6.33}
$$

All indeterminate momentum are integrated over $\int \frac{d^4k}{(2\pi)^3}$

.

Possible symmetry factor

Example 6.10. *The Feynman diagram*



, (6.34)

taking the value

$$\begin{aligned}
&\frac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} (2\pi)^4\delta(p_1+k-q)(2\pi)^4\delta(q-p_2-k) \\
&\times e^{ip_1x}e^{-ip_2x_2} \frac{i}{p_1^2-m^2+i\epsilon} \frac{i}{p_2^2-m^2+i\epsilon} \frac{i}{q^2-m^2+i\epsilon} \frac{i}{k^2-m^2+i\epsilon} \\
&= \frac{1}{2}(ig)^2 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \delta^{(4)}(q-p_2-q+p_1) \times \ldots
\end{aligned} \tag{6.35}$$

one delta function will always encode the conservation overall external momentum.

6.3.5 Momentum space Feynman rules for S matrix

Recall the LSZ formula

$$\begin{aligned}
&{}_{\text{out}}\langle p_{k+1}, \ldots, p_n | p_1, \ldots, p_k \rangle_{\text{in}} \\
&= \prod_{i=1}^{n} (-i)(p_i^2-m^2) \int d^4x_1 e^{ip_1x_1} \ldots \int d^4x_{k+1} e^{ip_{k+1}x_{k+1}} {}_{\text{out}}\langle \Omega | \mathscr{T}\left\{\phi(x_1)\ldots\phi(x_n)\right\} |\Omega\rangle_{\text{in}} \, .
\end{aligned} \tag{6.36}$$

All the external propagators are cancelled ("amputated").

All external momentum are put on-shell.

Example 6.11.

$$\begin{aligned}
{}_{out}\langle p_2 | p_1 \rangle_{in} = {} & (-i)(p_1^2-m^2)(-i)(p_2^2-m^2) \int d^4x_1 e^{-ip_1x_1} \int d^4x_2 e^{ipx_2} \times \\
& \frac{1}{2}(ig)^2 \int \frac{d^4\tilde{p}_1}{(2\pi)^4} \int \frac{d^4\tilde{p}_2}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} (2\pi)^4\delta^{(4)}(\tilde{p}_1-\tilde{p}_2) \frac{ie^{i\tilde{p}_1x_1}}{\tilde{p}_1^2-m^2+i\epsilon} \frac{ie^{-i\tilde{p}_2x_2}}{\tilde{p}_2^2-m^2+i\epsilon} \\
& \times \frac{i}{q^2-m^2+i\epsilon} \frac{i}{(q-\tilde{p}_1)^2-m^2+i\epsilon} \\
= {} & \frac{1}{2}(ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-m^2+i\epsilon} \frac{i}{(q-p_1)^2-m^2+i\epsilon} (2\pi)^4\delta^{(4)}(p_1-p_2) \, .
\end{aligned} \tag{6.37}$$

We set

$$i\mathscr{M} = \frac{1}{2}(ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-m^2+i\epsilon} \frac{i}{(q-p_1)^2-m^2+i\epsilon} \, , \tag{6.38}$$

from $S = 1 + iT = 1 + i\mathscr{M}(2\pi)\delta^{(4)}(\sum p_i)$.

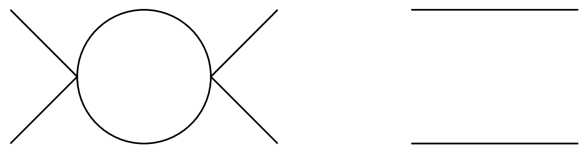


Figure 6.1: Different type of Feynman diagram

Summary of the Feynman rules for S -matrix computations

1. internal line contribute $\frac{i}{p^2-m^2+i\epsilon}$.
2. vertices yield factor of ig.
3. external lines carry on shell momentum into the diagram.
4. momentum conserved at each vertex.
5. integrate over undetermined momenta.

6.3.6 The connected S -matrix

The S-matrix from process $k \to n$: $p_i^1 \dots p_i^k \to p_f^1, \dots, p_f^n$ will generically have contributions from subprocess of the type

$$\begin{aligned} &p_i^{k_1} \dots p_i^{k_j} \to p_f^{n_1} \dots p_f^{n_l} \,, \\ &p_i^{\tilde{k}_1} \dots p_i^{\tilde{k}_j} \to p_f^{\tilde{n}_1} \dots p_f^{\tilde{n}_l} \,, \end{aligned} \tag{6.39}$$

i.e. product of process $\tilde{k} \to \tilde{n}$ for $\tilde{k} < k$ and $\tilde{n} < n$. At the level of Feynman diagrams, these are disconnected diagrams.

Example 6.12. *See figure (6.1).*

Such diagrams can be dropped if we are interested in the connect S-matrix.

Chapter 7

Equation of motion for quantum fields

In Chapter 2 we introduce the Euler-Lagrangian equation

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathscr{L}}{\partial \phi} = 0\,. \tag{7.1}$$

In QFT, by imposing canonical commutations, EoM for quantum fields leads to the same Feynman rules for evaluating VEVs of time order product as previously.

7.1 Differential operators and $\mathscr{T}$ -product

In this section we explore the relation

$$\Box_{x_1} \langle\Omega| \mathscr{T}\{\phi(x_1)\dots\phi(x_n)\} |\Omega\rangle \overset{?}{\leftrightarrow} \langle\Omega| \mathscr{T}\{\Box\phi(x_1)\dots\phi(x_n)\} |\Omega\rangle\ , \tag{7.2}$$

where we ignore the in/out subscript since we use the $|\Omega\rangle$ to denote the vacuum in the interacting theory. The spatial derivative part is trivial. The thing that needs consideration is the time derivative part,

$$\begin{aligned}
&\partial_t \langle\Omega| \mathscr{T}\left\{\phi(x)\phi(x')\right\} |\Omega\rangle = \partial_t \left(\langle\Omega| \phi(x)\phi(x') |\Omega\rangle \Theta(t-t') + \langle\Omega| \phi(x')\phi(x) |\Omega\rangle \Theta(t'-t) + \right) \\
&= \langle\Omega| \mathscr{T}\left\{\partial_t\phi(x)\phi(x')\right\} |\Omega\rangle + \langle\Omega| \phi(x)\phi(x') |\Omega\rangle \delta(t-t') - \langle\Omega| \phi(x')\phi(x) |\Omega\rangle \delta(t-t') \\
&= \langle\Omega| \mathscr{T}\left\{\partial_t\phi(x)\phi(x')\right\} |\Omega\rangle + \langle\Omega| [\phi(x),\phi(x')] |\Omega\rangle \delta(t-t') \\
&= \langle\Omega| \mathscr{T}\left\{\partial_t\phi(x)\phi(x')\right\} |\Omega\rangle\ ,
\end{aligned} \tag{7.3}$$

because $[\phi(x),\phi(x')] = 0$.

$$\begin{aligned}
&\partial_t^2 \langle\Omega| \mathscr{T}\left\{\phi(x)\phi(x')\right\} |\Omega\rangle = \partial_t \langle\Omega| \mathscr{T}\left\{\partial_t\phi(x)\phi(x')\right\} |\Omega\rangle = \\
&\quad \langle\Omega| \mathscr{T}\left\{\partial_t^2\phi(x)\phi(x')\right\} |\Omega\rangle + \langle\Omega| [\partial_t\phi(x),\phi(x')] |\Omega\rangle \delta(t-t') \\
&\quad = \langle\Omega| \mathscr{T}\left\{\partial_t^2\phi(x)\phi(x')\right\} |\Omega\rangle - i\delta^{(4)}(x-x')\,,
\end{aligned} \tag{7.4}$$

where we have used the commutation relation $[\partial_t\phi(x),\phi(x')] = -i\delta^{(3)}(\vec{x}-\vec{x'})$. We called the second term $-i\delta^{(4)}(x-x')$ as the contact term.

$$\Rightarrow \quad \Box_x \langle\Omega| \mathscr{T}\left\{\phi(x)\phi(x')\right\} |\Omega\rangle = \langle\Omega| \mathscr{T}\left\{\Box\phi(x)\phi(x')\right\} |\Omega\rangle - i\delta^{(4)}(x-x')\,. \tag{7.5}$$

By reduction, we have the following formula

$$\Box_x \langle\Omega| \mathscr{T}\{\phi(x)\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \langle\Omega|\mathscr{T}\{\Box\phi(x)\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle$$
$$-\sum_{i=1}^{n} i\delta^{(4)}(x-x_i)\langle\Omega|\mathscr{T}\left\{\phi(x_1)\dots\phi(x_{i-1})\widehat{\phi(x_i)}\phi(x_{i+1})\dots\phi(x_n)\right\}|\Omega\rangle\ . \tag{7.6}$$

7.2 Schwinger Dyson equations

The Lagrangian of the real scalar field is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \mathscr{L}_{\text{int}}(\phi)\,, \tag{7.7}$$

whose EoM is

$$(\Box_x + m^2)\phi(x) - \frac{\partial\mathscr{L}_{\text{int}}}{\partial\phi} = 0\,. \tag{7.8}$$

Combine it with the equation (7.6) we have

$$(\Box_x + m^2)\langle\Omega|\mathscr{T}\{\phi(x)\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \langle\Omega|\mathscr{T}\left\{\frac{\partial\mathscr{L}_{\text{int}}}{\partial\phi}\phi(x)\phi(x_1)\dots\phi(x_n)\right\}|\Omega\rangle$$
$$-i\sum_{i=1}^{n}\delta^{(4)}(x-x_i)\langle\Omega|\mathscr{T}\left\{\phi(x_1)\dots\phi(x_{i-1})\widehat{\phi(x_i)}\phi(x_{i+1})\dots\phi(x_n)\right\}|\Omega\rangle\ , \tag{7.9}$$

which is known as Schwinger-Dyson equation.

7.3 Evaluating VEVs of T-products in free theory via Schwinger-Dyson equation

Take the two-point function as an example. In free case, the Schwinger-Dyson equation reduces to

$$(\Box_x + m^2)\langle 0|\mathscr{T}\left\{\phi(x)\phi(x')\right\}|0\rangle = -i\delta^{(4)}(x-x')\,, \tag{7.10}$$

which is the equation satisfied by Green function of the operator $\Box_x^2 + m^2$. For n-point function, the strategy is

- The operator $(\Box_x^2 + m^2)$ action on n-point function reduces the number of fields.
- introduce that operators by involving our result for 2-point function, then integrate by parts.

Example 7.1. *For four-point function, rewriting as*

$$
\begin{aligned}
&\langle 0|\mathscr{T}\{\phi(x_1)\dots\phi(x_4)\}|0\rangle=\int d^4x\delta^{(4)}(x-x_1)\langle 0|\mathscr{T}\{\phi(x)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle\\
&=\int d^4xi(\Box_x^2+m^2)D_F(x-x_1)\langle 0|\mathscr{T}\{\phi(x)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle\\
&=i\int d^4xD_F(x-x_1)(\Box_x^2+m^2)\langle 0|\mathscr{T}\{\phi(x)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle\\
&=\int d^4xD_F(x-x_1)\Big(\delta^{(4)}(x-x_2)\langle 0|\mathscr{T}\left\{\phi(x_3)\phi_(x_4)\right\}|0\rangle\\
&+\delta^{(4)}(x-x_3)\langle 0|\mathscr{T}\left\{\phi(x_2)\phi_(x_4)\right\}|0\rangle+\delta^{(4)}(x-x_4)\langle 0|\mathscr{T}\left\{\phi(x_2)\phi_(x_2)\right\}|0\rangle\Big)\\
&=D_F(x_1-x_2)D_F(x_3-x_4)+D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3)\,.
\end{aligned}
\tag{7.11}
$$

7.4 Evaluating n-point function in interacting theory via Schwinger-Dyson equation

The strategy is the same as in free case. Now procedures leads also to terms with increased number of fields due to the $\mathscr{L}_{\text{int}}$ term.

Example 7.2. *2 point function in ϕ^3 theory, where $\mathscr{L}_{int}=\frac{g}{3!}\phi^3$.*

$$
\begin{aligned}
\langle\Omega|\mathscr{T}\{\phi(x_1)\phi(x_2)\}|\Omega\rangle&=\int d^4x\delta^{(4)}(x-x_1)\langle\Omega|\mathscr{T}\{\phi(x)\phi(x_2)\}|\Omega\rangle\\
&=\int d^4xi(\Box_x+m^2)D_F(x-x_1)\langle\Omega|\mathscr{T}\{\phi(x)\phi(x_2)\}|\Omega\rangle\\
&=i\int d^4xD_F(x-x_1)\left(\langle\Omega|\mathscr{T}\left\{\frac{\partial\mathscr{L}_{int}}{\partial\phi}\phi(x_2)\right\}|\Omega\rangle-i\delta^{(4)}(x-x_2)\right)\,.
\end{aligned}
\tag{7.12}
$$

Contact terms and LSZ

We have shown that

$$
{}_{\text{out}}\langle f|i\rangle_{\text{in}}=i\int d^4x_1e^{ip_1x_1}\dots i\int d^4xe^{ip_nx_n}{}_{\text{out}}\langle\Omega|\mathscr{T}\left\{(\Box_x+m^2)\phi(x_1)\dots\phi(\Box_x+m^2)\phi(x_n)\right\}|\Omega\rangle_{\text{in}}\,,\tag{7.13}
$$

where the red part is

$$
\begin{aligned}
&(\Box_x+m^2)\dots(\Box_x+m^2)_{\text{out}}\langle\Omega|\mathscr{T}\{(\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle_{\text{in}}\\
&+\text{terms involving }\delta\text{ function }\delta^{(4)}(x_i-x_j)\text{ multiplying}\\
&\text{time ordered produce of fields independent of }x_i\text{ and }x_j\,,
\end{aligned}
\tag{7.14}
$$

which give rise to the contributions

$$
\int d^4x_i\int d_j^4e^{\pm ip_ix_i}e^{\pm ip_jx_j}\delta^{(4)}(x_i-x_j)=\delta^{(4)}(p_i\pm p_j)\,,\tag{7.15}
$$

where

$$\begin{cases} + : \text{both outgoing or incoming, vanishes as } p_{i,j} > 0 \\ - : \text{one incoming, one outgoing, trivial scattering process, } p_i = p_j, \text{ excluded in our analysis.} \end{cases} \tag{7.16}$$

This analysis tells us we can drop the additional terms as claimed (case of $p_i = p_j$ requires further reasoning.)

Chapter 8

Irreducible unitary representations of the Poincaré group

8.1 The Poincaré group $\mathscr{P}_0$ as a symmetry of fundamental interactions

The inhomogeneous (full) Poincaré group is

$$\mathscr{P} = \mathbb{R}^4 \bowtie O(1,3)\,, \tag{8.1}$$

where $\mathbb{R}^4$ represents the translation $x \to x + a$ and $O(1,3)$ is the Lorentz group of all linear transformation of $\mathbb{R}^4$ that preserves the Minkowski metric. The multiplication rule and the inverse are

$$\begin{aligned} (a, S) \circ (b, T) &= (a + Sb, S \circ T) \\ (a, S)^{-1} &= (-S^{-1}a, S^{-1})\,. \end{aligned} \tag{8.2}$$

$O(1,3)$ has several components, we will require that

$$\mathscr{P}_0 = \mathbb{R}^4 \bowtie SO^{\uparrow}(1,3)\,, \tag{8.3}$$

where $SO^{\uparrow}$ is proper orthochrous Lorentz group, which is a connected component of group $O(1,3)$. It containing the identity. In particular, timer reversal T, space inversion P in this subgroup.

8.2 Wigner's theorem and Bargmann's theorem

Physical states are associated to rays, i.e. element of $\mathscr{H}$ up to phase. The space of rays is denoted as $\mathscr{R} = \mathscr{H}/N$ where N is the equivalence relation: for $\phi, \psi \in \mathscr{H}$, $\phi \sim \psi \iff \exists\, \theta \in [0, 2\pi), \phi = e^{i\theta}\psi$. Recall that given $r, r' \in \mathscr{R}$, transition probability P from r to r' is independent of choice of representations. $P = |\langle \phi' | \phi \rangle|^2$ for any $\phi, \phi' \in \mathscr{H}$ such that $[\phi] = r, [\phi'] = r$.

Definition 8.1. *$\mathscr{P}_0$ is a symmetry means that for any element of $\mathscr{P}_0$ gives rise to a map $\mathscr{R} \to \mathscr{R}$, which preserves probabilities, i.e. given $g \in \mathscr{P}_0$, $\exists\, T(g) : \mathscr{R} \to \mathscr{R}$ such that for $r, r' \in \mathscr{R}$,*

$$|\langle r | r' \rangle|^2 = |\langle T(g) r | T(g) r' \rangle|^2\,. \tag{8.4}$$

Theorem 8.1. *Wigner's theorem: Any bijection $\mathscr{R} \to \mathscr{R}$ which conserves the transition probabilities of uni rays can be represented by an operator on $\mathscr{H}$ that is either*

1. *linear and unitary*

2. *anti-linear and anti-unitary, i.e. $U(\xi |\phi\rangle + \eta |\psi\rangle) = \bar{\xi} U |\phi\rangle + \bar{\eta} U |\psi\rangle$ (anti-linear), $\langle U\phi | U\psi\rangle = \langle \phi \bar{|} \psi\rangle$ (anti-unitary).*

Remark 8.1. *Symmetries connected continuously to the identity (such as rotations, element of $SO^{\uparrow}(1,3)$) must be represented by linear unitary operator because the identity is linear and unitary.*

Given $g_1, g_2 \in \mathscr{P}_0$ acting with $U(g_2)U(g_1)$ or with $U(g_1 \circ g_2)$ should give rise to the same physical state, which means that $U(g_2)U(g_1) = e^{i\phi(g_1,g_2)}U(g_2 \circ g_1)$. We conclude that $\mathscr{P}_0$ must be projectively represented on $\mathscr{H}$.

Theorem 8.2. *Bargmann's theorem: Every projective representation of the universal cover $\hat{\mathscr{P}}_0$ of $\mathscr{P}_0$ can be lifted to a unitary representation of $\hat{\mathscr{P}}$.*

This theorem tell us that $\mathscr{H}$ should furnish unitary representation U of $\hat{\mathscr{P}}_0$. i.e.

$$\begin{aligned} U : \hat{\mathscr{P}}_0 &\to \mathrm{Aut}(\mathscr{H}) \\ (a, \Lambda) &\mapsto U(a, \Lambda)\,, \end{aligned} \tag{8.5}$$

such that $U\left((a_2, \Lambda_2) \circ (a_1, \Lambda_1)\right) = U(a_2, \Lambda_2) \circ U(a_1, \Lambda_1)$.

8.3 Particles

$\mathscr{H}$ can be decomposed into eigenspace with respect to a maximal set of mutually commuting operators (including $H, \vec{P}$). Let $|\psi\rangle$ lie in such an eigenspace (in particular $P^\mu |\psi\rangle = p^\mu |\psi\rangle$). In our definition of particle, $|\psi\rangle$ and $U(a,\Lambda)|\psi\rangle$ should not correspond to different particles, which guides us to study irreducible representation of $\hat{\mathscr{P}}_0$.

Definition 8.2. *The subspace $\mathscr{H}_i \subset \mathscr{H}$ is irreducible if*

- *it is invariant under action of $\hat{\mathscr{P}}_0$.*
- *no proper subset of $\mathscr{H}_i$ other than $\{0\}$ has above property.*

8.4 The representation of spacetime translation

The spacetime translation can be represented as

$$U(a, \mathbb{1}) = e^{i a_\mu \hat{p}^\mu}\,, \hat{p}^\mu = (H, \vec{p}). \tag{8.6}$$

We can see that $\hat{p}^\mu$ is self-adjoint, which means that we can diagonalize the Hilbert space $\mathscr{H}$ as $\mathscr{H} = \bigoplus V_{\vec{p}}$, where $V_{\vec{p}}$ is the eigensubspace labeled by spatial momentum $\vec{p}$. Let $|\vec{p}\rangle \in V_{\vec{p}}$, $\langle |\vec{p}\rangle\rangle$ is a irreducible representation of spacetime translation. The action

$$U(a, \mathbb{1})|\vec{p}\rangle = e^{iap}|\vec{p}\rangle \quad \Rightarrow \quad \langle |\vec{p}\rangle\rangle \text{ is closed under the action of } U(a, \mathbb{1}).$$

The fact that irreps (irreducible representations) are one dimensional follows from the theorem that complex irreps of Abelian group is one dimensional. As we can decompose any element of $(a, \Lambda) \in \mathscr{P}_0$ as $(a, \Lambda) = (a, 1) \circ (0, \Lambda)$, we can focus on the spacetime rotation $U(\Lambda) := U(0, \Lambda)$.

8.5 Unitary representation of the Lorentz group via Wigner's method of induced representations

Theorem 8.3. *The only finite dimensional unitary representation of the Lorentz group is the trivial representation.*

Example 8.1. *Consider the 4 vector representation, the Lorentz boost along the x axis takes the following form,*

$$A = \begin{pmatrix} \cosh\beta x & \sinh\beta x & & \\ \sinh\beta x & \cosh\beta x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{8.7}$$

however, $A^\dagger A \neq \mathbb{1}$, A is not a unitary matrix.

The method of induced representation (Wigner)

The strategy is

1. study a subgroup (called the little group) of the Lorentz group which does have finite dimensional representations.
2. lift these representations to the full Lorentz group, which is so called the induced representation.

In physics, we are interested little group H_m^k. For $m > 0$, choose the momentum $k^\mu \in \mathbb{R}^4$ such that $k^\mu k_\mu = m^2$ and $k^0 > 0$, we define H_m^k as

$$H_m^k = \{\Lambda \in SO^\uparrow(1,3) | \Lambda^\mu{}_\nu k^\nu = k^\mu\}, \tag{8.8}$$

which is the set of the spacetime rotation keeping the four momentum k^μ invariant. It is easy to verify it is indeed a subgroup of $SO^\uparrow(1,3)$. Physically, m will correspond to the mass of the particle, k is its four momentum. We begin our discussion by considering momentum eigensubspace of $\mathscr{H}$ with momentum k and $m > 0$. A convenient choice is the particle at rest: $k = (m, 0, 0, 0)$. Obviously the little group is $H_m^{(m,0,0,0)} = SO(3)$, because only boost can generate 3-momentum. Since the little group is now $SO(3)$, its double cover is $SU(2)$. The irreducible representation of $SU(2)$is classified by non-negative half-integers or integers $J \in \frac{1}{2}\mathbb{N} \cup \{0\}$, which is so called the spin of $(2J+1)$ representation. If we restrict $J \in \mathbb{N}$, actually it is the irreps of $SO(3)$. A state transforming in the representation J of $SU(2)$carries an index $\sigma \in \{-J, \dots, J\}$. We denoted as $|\psi_{k,\sigma}\rangle$ such that $\hat{p}^\mu |\psi_{k,\sigma}\rangle = k^\mu |\psi_{k,\sigma}\rangle$. For two rotations $R_1, R_2 \in H_m^k$, because

$$
\begin{aligned}
U(R_1)\left|\psi_{k,\sigma}\right\rangle &= \sum_{\sigma'=-J}^{J} D(R_1)_{\sigma'\sigma}\left|\psi_{k,\sigma'}\right\rangle \\
U(R_2)U(R_1)\left|\psi_{k,\sigma}\right\rangle = U(R_2)\left(\sum_{\sigma'=-J}^{J} D(R_1)_{\sigma'\sigma}\left|\psi_{k,\sigma'}\right\rangle\right) &= \sum_{\sigma'=-J}^{J}\sum_{\sigma''=-J}^{J} D(R_1)_{\sigma'\sigma}D(R_2)_{\sigma''\sigma'}\left|\psi_{k,\sigma''}\right\rangle . \\
&= \sum_{\sigma''=-J}^{J} D(R_2\circ R_1)_{\sigma\sigma''}\left|\psi_{k,\sigma''}\right\rangle = U(R_2\circ R_1)\left|\psi_{k,\sigma}\right\rangle \ .
\end{aligned} \tag{8.9}
$$

Note that $D^\dagger(R)D(R) = \mathbb{1}$ as the irreps are unitary. We finish the first step of the strategy.

Next, for any $p \in \mathbb{R}^4$ such that $p^2 = m^2$ for a fiducial Lorentz transformation $L(p)$,

$$
L(p)^\mu{}_\nu k^\nu = p^\mu \ . \tag{8.10}
$$

Define $\left|\psi_{p,\sigma}\right\rangle = U(L(p))\left|\psi_{k,\sigma}\right\rangle$. $\left|\psi_{p,\sigma}\right\rangle$ is a momentum eigenstate as

$$
\begin{aligned}
U(a,\mathbb{1})\left|\psi_{p,\sigma}\right\rangle &= U(a,\mathbb{1})U(0,L(p))\left|\psi_{k,\sigma}\right\rangle \\
&= U(a,L(p))\left|\psi_{k,\sigma}\right\rangle \\
&= U(0,L(p))U(L^{-1}(p)a,\mathbb{1})\left|\psi_{k,\sigma}\right\rangle \\
&= U(L(p))e^{i(L^{-1}(p)a,k)}\left|\psi_{k,0}\right\rangle \\
&= e^{i(a,L(p)k)}\left|\psi_{p,\sigma}\right\rangle \ ,
\end{aligned} \tag{8.11}
$$

where $(\cdot,\cdot)$ is the Minkowski inner product and we used the multiplication rule to extract the result. We conclude that the subspace

$$
\left\langle \{\left|\psi_{p,\sigma}\right\rangle | p \in \mathbb{R}^4 \,, p^2 = m^2 \,, p^0 > 0 \,, \sigma \in \{-J,\dots,J\}\}\right\rangle \ , \tag{8.12}
$$

furnishes a unitary irrep of $\mathscr{P}_0$,

1. $U(a,1)\left|\psi_{p,\sigma}\right\rangle = e^{iap}\left|\psi_{p,\sigma}\right\rangle$.

2.
$$
\begin{aligned}
U(\Lambda)\left|\psi_{p,\sigma}\right\rangle &= U(\Lambda)U(L(p))\left|\psi_{k,\sigma}\right\rangle \\
&= U(L(\Lambda p))U(L^{-1}(\Lambda p))U(\Lambda L(p))\left|\psi_{k,\sigma}\right\rangle \\
&= U(L(\Lambda p))W(\Lambda,p)\left|\psi_{k,\sigma}\right\rangle \ ,
\end{aligned} \tag{8.13}
$$

where we define $W(\Lambda,p) := U(L^{-1}(\Lambda p))U(\Lambda L(p))$ which maps the momentum $k \to p \to \Lambda p \to k$, then $W(\Lambda,p) \in H_m^k$. From the previous discussion, using that conclusion that

$$
W(\Lambda,p)\left|\psi_{k,\sigma}\right\rangle = \sum_{\sigma'=-J}^{J} D(W(\Lambda,p))_{\sigma'\sigma}\left|\psi_{k,\sigma'}\right\rangle \ , \tag{8.14}
$$

we can write

$$U(\Lambda)\left|\psi_{p,\sigma}\right\rangle = U(L(\Lambda,p))\sum_{\sigma'=-J}^{J} D(W(\Lambda,p))_{\sigma'\sigma}\left|\psi_{k,\sigma'}\right\rangle \;=\; \sum_{\sigma'=-J}^{J} D(W(\Lambda,p))_{\sigma'\sigma}\left|\psi_{\Lambda p,\sigma'}\right\rangle . \tag{8.15}$$

It is easy to check that under this definition $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$.

Remark 8.2. *We can see that through this way we could introduce the spin indices naturally and the spin degrees of freedom naturally separate with the momentum (spacetime) degrees of freedom. The representation theory shows its strong power again.*

Theorem 8.4. *This representation is irreducible.*

Proof: The proof is direct because of the definition of the coefficient $D(R)$.

Remark 8.3. *The representation space* $\left\langle\{\left|\psi_{p,\sigma}\right\rangle | p\in\mathbb{R}^4\,, p^2=m^2\,, p^0>0\,,\sigma\in\{-J,\ldots,J\}\}\right\rangle$ *is infinite dimensional.*

We can introduce an inner product on the representation space via

$$\left\langle\psi_{p,\sigma}\middle|\psi_{p',\sigma'}\right\rangle = (2\pi)^3 2\omega_p \delta^{(3)}(\vec{p}-\vec{p}')\delta_{\sigma\sigma'} . \tag{8.16}$$

We need to check if such an inner product is Lorentz invariant. Consider

$$\begin{aligned} \delta^{(4)}(p-p') &= \delta(p^0-\sqrt{m+\vec{p}^2})\delta^{(3)}(\vec{p}-\vec{p}') \\ &= 2p^0\delta(p^2-m^2)\theta(p^0)\delta^{(3)}(\vec{p}-\vec{p}') \end{aligned} \tag{8.17}$$

From equation (1.27), we know that the invariant measure is

$$\int \frac{d^3\vec{p}}{\sqrt{\vec{p}^2+m^2}} . \tag{8.18}$$

For the three dimensional delta function, we define it by

$$F(\vec{p}) = \int F(\vec{p}')\delta^{(3)}(\vec{p}-\vec{p}')d^3\vec{p}' , \tag{8.19}$$

this guides us that we could define the so called invariant delta function under the Lorentz transformation as

$$\sqrt{\vec{p}^2+m^2}\delta^{(3)}(\vec{p}-\vec{p}') = \sqrt{(\Lambda\vec{p})^2+m^2}\delta^{(3)}(\Lambda\vec{p}-\Lambda\vec{p}') , \tag{8.20}$$

$$\Rightarrow \quad 2p^0\delta^{(3)}(\vec{p}-\vec{p}') = 2(\Lambda p)^0\delta^{(3)}(\Lambda\vec{p}-\Lambda\vec{p}') . \tag{8.21}$$

We then check if such an representation is unitary,

$$
\begin{aligned}
\langle U(\Lambda)\psi_{p_1,\sigma_1}|U(\Lambda)\psi_{p_2,\sigma_2}\rangle &= \sum_{\sigma_1',\sigma_2'=-J}^{J} D^*(W(\Lambda,p_1))_{\sigma_1'\sigma_1} D(W(\Lambda,p_2))_{\sigma_2'\sigma_2} \left\langle \psi_{\Lambda p_1,\sigma_1'} \middle| \psi_{\Lambda p_2,\sigma_2'} \right\rangle \\
&= \sum_{\sigma_1',\sigma_2'=-J}^{J} D^*(W(\Lambda,p_1))_{\sigma_1'\sigma_1} D(W(\Lambda,p_2))_{\sigma_2'\sigma_2} (2\pi)^3 2(\Lambda p_1)^0 \delta^{(3)}(\Lambda\vec{p}_1 - \Lambda\vec{p}_2)\delta_{\sigma_1'\sigma_2'} \\
&= \sum_{\sigma=-J}^{J} D^*(W(\Lambda,p_1))_{\sigma\sigma_1} D(W(\Lambda,p_2))_{\sigma\sigma_2} (2\pi)^3 2p^0 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\,,
\end{aligned}
\tag{8.22}
$$

by using the unitary of the D matrix, $\sum_{\sigma=-J}^{J} D^*(W(\Lambda,p_1))_{\sigma\sigma_1} D(W(\Lambda,p_2))_{\sigma\sigma_2} = \delta_{\sigma_1\sigma_2}$, we conclude that

$$
\langle U(\Lambda)\psi_{p_1,\sigma_1}|U(\Lambda)\psi_{p_2,\sigma_2}\rangle = \langle \psi_{p_1,\sigma_1}|\psi_{p_2,\sigma_2}\rangle \ , \tag{8.23}
$$

representation $U(\Lambda)$ is indeed a unitary representation.

8.6 Transformation of free creation and annihilation operators

Recall that $a^\dagger(p,\sigma)\,|0\rangle = \frac{1}{\sqrt{2\omega_p}}\,|\vec{p},\sigma\rangle = \frac{1}{\sqrt{2\omega_p}}\,|\psi_{p,\sigma}\rangle$. The transformation of $a^\dagger$ follows from that of $\psi_{\vec{p},\sigma}$. We assume that the vacuum is Lorentz invariant

$$
U(\Lambda)\,|0\rangle = |0\rangle \ . \tag{8.24}
$$

The transformation of $|\psi\rangle_{p,\sigma}$,

$$
\begin{aligned}
U(\Lambda)\,|\psi_{p,\sigma}\rangle &= \sum_{\sigma'=-J}^{J} D(W(\Lambda,p))_{\sigma'\sigma}\,\left|\psi_{\Lambda\vec{p},\sigma}\right\rangle \\
\Rightarrow \quad \sqrt{2\omega_p}U(\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(\Lambda) &= \sqrt{2\omega_{\Lambda p}} \sum_{\sigma'=-J}^{J} D(W(\Lambda,p))_{\sigma'\sigma} a^\dagger(\Lambda\vec{p},\sigma') \\
\Rightarrow \quad U(\Lambda)a^\dagger(p,\sigma)U^{-1}(\Lambda) &= \sqrt{\frac{2\omega_{\Lambda p}}{2\omega_p}} \sum_{\sigma'=-J}^{J} D(W(\Lambda,p))_{\sigma'\sigma} a^\dagger(\Lambda p,\sigma') \\
\Rightarrow \quad U(\Lambda)a(p,\sigma)U^{-1}(\Lambda) &= \sqrt{\frac{2\omega_{\Lambda p}}{2\omega_p}} \sum_{\sigma'=-J}^{J} D^*(W(\Lambda,p))_{\sigma'\sigma} a(\Lambda p,\sigma')
\end{aligned}
\tag{8.25}
$$

8.7 Transformation of free massive scalar quantum field

The scalar representation is spin $J=0$ representation of $SO(3)$, the D matrix

$$
D_{00}(R) = \mathbb{1} \quad \Rightarrow \quad \text{can drop the } \sigma \text{ index}\,.
$$

The scalar field constructed by a and $a^\dagger$ is

$$\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right) \\
U(\Lambda)\phi(x)U^{-1}(\Lambda) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{\sqrt{2\omega_{\Lambda p}}}{\omega_p}\left(a_{\Lambda p} e^{-ipx} + a_{\Lambda p}^\dagger e^{ipx}\right) \\
&\overset{\tilde{p}=\Lambda p}{=} \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{\sqrt{2\omega_{\tilde{p}}}}{2\omega_{\tilde{p}}}\left(a_{\tilde{p}} e^{-i(\Lambda^{-1}\tilde{p},x)} + a_{\tilde{p}}^\dagger e^{i(\Lambda^{-1}\tilde{p},x)}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-i(p,\Lambda x)} + a_p^\dagger e^{i(p,\Lambda x)}\right) = \phi(\Lambda x)\,,
\end{aligned} \tag{8.26}$$

where we have used the property of the invariant measure. This is the first important result, we get the transformation property of the scalar field compatible with the Lorentz invariant theory.

8.8 Construction of massive vector field

8.8.1 Objective and Ansatz

Objective: construct a quantum vector field $A^\mu(x)$ such that

$$U(\Lambda)A^\mu(x)U^{-1}(\Lambda) = \mathscr{D}(\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x)\,. \tag{8.27}$$

Remark 8.4. *Pay attention the notation here, D is the representation of little group H_m^k, and $\mathscr{D}$ is the finite dimensional representation of $SO^\uparrow(1,3)$.*

The group structure requires

$$U(\Lambda_1)U(\Lambda_2)A^\mu(x)U(\Lambda_2^{-1})U(\Lambda_1^{-1}) = \mathscr{D}\left[(\Lambda_1\Lambda_2)^{-1}\right]^\mu{}_\rho A^\rho(\Lambda_1\Lambda_2 x)\,. \tag{8.28}$$

Remark 8.5. *The fields are not constrained to transform in unitary representations.*

What type of particles should A^μ package? A natural choice is the spin 1 ($J=1$) massive particle.

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{\sigma=-1}^{+1}\left(\varepsilon^\mu{}_\sigma(p)a_{p,\sigma}e{-ipx} + \varepsilon^\mu{}_\sigma{}^*(p)a_{p,\sigma}^\dagger e^{ipx}\right)\,, \tag{8.29}$$

where $\varepsilon^\mu{}_\nu$ is the polarization tensor and the conjugate in the second term is to make the field Hermitian. We will show what is the requirement of the polarization tensor should satisfy in the later discussion.

8.8.2 What property do the function $\varepsilon^\mu{}_\sigma(p)$ have to satisfy?

Our goal is that

$$U(\Lambda)A^\mu(x)U^{-1}(\Lambda) = \mathscr{D}(\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x)\,, \tag{8.30}$$

putting the Ansatz into the equation, and recall the transformation property of a and $a^\dagger$,

$$
\begin{aligned}
U(\Lambda)A^\mu(x)U^{-1}(\Lambda) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{\sigma=-1}^{+1}\left[\varepsilon^\mu{}_\sigma U(\Lambda)a_{p,\sigma}U^{-1}(\Lambda)e^{-ipx}+\varepsilon^\mu{}_\sigma{}^*(p)U(\Lambda)a^\dagger_{p,\sigma}U^{-1}(\Lambda)e^{ipx}\right]\\
&= \frac{d^3p}{(2\pi)^3}\frac{\sqrt{2\omega_{\Lambda p}}}{2\omega_p}\sum_{\sigma,\sigma'}\left[\varepsilon^\mu{}_\sigma(p)D^*_{\sigma'\sigma}(W(\Lambda,p))a_{\Lambda p,\sigma'}e^{-ipx}\right.\\
&\left.+\varepsilon^\mu{}_\sigma{}^*(p)D_{\sigma'\sigma}(W(\Lambda,p))a^\dagger_{\Lambda p,\sigma'}e^{ipx}\right]\\
&= \mathscr{D}(\Lambda^{-1})^\mu{}_\nu\int\frac{d^3p}{(2\pi)^3}\frac{\sqrt{2\omega_p}}{2\omega_p}\left[\varepsilon^\nu{}_\sigma(p)a_{p,\sigma}e^{-ip\Lambda x}+\varepsilon^\nu{}_\sigma{}^*(p)a^\dagger_{p,\sigma}e^{ip\Lambda x}\right]\\
&\overset{\tilde p=\Lambda^{-1}p}{=} \mathscr{D}(\Lambda^{-1})^\mu{}_\nu\int\frac{d^3\tilde p}{(2\pi)^3}\frac{\sqrt{\omega_{\Lambda\tilde p}}}{2\omega_{\tilde p}}\left[\varepsilon^\nu{}_\sigma(\Lambda\tilde p)a_{\Lambda\tilde p,\sigma}e^{-i\tilde px}+\varepsilon^\nu{}_\sigma{}^*(\Lambda\tilde p)a^\dagger_{\Lambda\tilde p,\sigma}e^{i\tilde px}\right]\,.
\end{aligned}
\tag{8.31}
$$

Compare the two sides of the equation, we have

$$
\sum_\sigma D^*_{\sigma'\sigma}(W(\Lambda,p))\varepsilon^\mu{}_\sigma(p)=\sum_\nu\mathscr{D}(\Lambda^{-1})^\mu{}_\nu\varepsilon^\nu{}_{\sigma'}(\Lambda p)\,. \tag{8.32}
$$

Pay attention about the position of σ'. We split this relation in two steps:

1. Note that $\varepsilon^\mu{}_\sigma(k)$ fixes $\varepsilon^\mu{}_\sigma(p)$ as we could choose Λ such that $\Lambda p=k\Rightarrow\Lambda=L^{-1}(p)$. Based on the definition of W

$$
\begin{aligned}
&W(\Lambda,p)=L(\Lambda p)^{-1}\Lambda L(p)\quad\Rightarrow\quad W(L^{-1}(p),p)=L\left(L^{-1}(p)p\right)^{-1}L^{-1}(p)L(p)=\mathbb{1}\,,\\
&\Rightarrow\quad\sum_\sigma D^*_{\sigma'\sigma}(\mathbb{1})\varepsilon^\mu{}_\sigma(p)=\sum_\sigma\delta_{\sigma'\sigma}\varepsilon^\mu{}_\sigma(p)=\varepsilon^\epsilon{}_{\sigma'}(p)=\sum_\nu\mathscr{D}(L(p))^\mu{}_\nu\varepsilon^\nu{}_{\sigma'}(k)
\end{aligned}
\tag{8.33}
$$

2. Derive the relation for $\varepsilon^\mu{}_\sigma(k)$: we choose $p=k$, $\Lambda=\rho\in H^k_m$,

$$
\begin{aligned}
W(\rho,k)=L(\rho k)^{-1}\rho L(k)\rho\,,\Rightarrow\quad&\sum_\sigma D^*_{\sigma'\sigma}(\rho)\varepsilon^\mu{}_\sigma(k)=\sum_\nu\mathscr{D}(\rho^{-1})^\mu{}_\nu\varepsilon^\nu{}_{\sigma'}(\rho k)\\
&\sum_\sigma D^\dagger_{\sigma\sigma'}(\rho)\varepsilon^\mu{}_\sigma(k)=\sum_\nu\mathscr{D}(\rho^{-1})^\mu{}_\nu\varepsilon^\nu{}_{\sigma'}(k)\,.
\end{aligned}
\tag{8.34}
$$

The claim 1 and 2 is equivalent to the equation of ε (8.32).

Remark 8.6. *The condition (8.32) as well as the equivalence (1,2) hold for all $\mathscr{D}$, which is the finite dimensional (non unitary for $\mathscr{D}$ non-trivial) irrep of $SO^\uparrow(1,3)$ under which the quantum transforms. And D is the finite dimensional unitary irrep of H^k_m under which the states transform.*

8.8.3 Solving the constraints on $\varepsilon^\mu{}_\sigma(k)$ for massive vector field

By using the unitarity of D, we have $D^\dagger_{\sigma\sigma'}(\rho)=D(\rho^{-1})_{\sigma\sigma'}$, the relation (8.32) is

$$
\sum_\sigma\varepsilon^\mu{}_{\sigma'}(k)D(\rho^{-1})_{\sigma\sigma'}=\sum_\nu\mathscr{D}(\rho^{-1})^\mu{}_\nu\varepsilon^\nu{}_{\sigma'}(k)\,. \tag{8.35}
$$

D matrix is 3×3 matrix, while $\mathscr{D}$ matrix is 4 matrix, which requires that ε is a 4×3 matrix with 4 spacetime indices and 3 spin indices, setting

$$\varepsilon(k) = \begin{bmatrix} a & b & c \\ & A_{3\times 3} & \end{bmatrix} . \tag{8.36}$$

Remember that we are now considering the massive vector field with four momentum $k = (m, 0, 0, 0)$, the element ρ should keep the time component invariant, which means that the $\mathscr{D}$ matrix can be written as

$$\mathscr{D}(\rho^{-1}) = \left[\begin{array}{c|c} 1 & \\ \hline & \mathfrak{D}(\rho^{-1})_{3\times 3} \end{array}\right] . \tag{8.37}$$

The relation expressed in matrix notation is

$$\varepsilon(k)D(\rho^{-1}) = \mathscr{D}(\rho^{-1})\epsilon(k) = \begin{bmatrix} a & b & c \\ & \mathfrak{D}(\rho^{-1})A & \end{bmatrix} , \tag{8.38}$$

such a relation holds for arbitrary $\rho \in SO(3)$. Notice that the spin-1 representation for the little group H_m^k is the same as $SO(3)$ representation, which means that $D(\rho) = \mathfrak{D}(\rho)$, in this condition, the relation of $\varepsilon(k)$ can be written as

$$\varepsilon(k)D(\rho^{-1}) = \mathscr{D}(\rho^{-1})\varepsilon(k) = \begin{bmatrix} a & b & c \\ & D(\rho^{-1})A & \end{bmatrix} . \tag{8.39}$$

Since this relation holds for all $\rho \in SO(3)$, we consider the infinitesimal transformation. The infinitesimal rotation around x-axis and z-axis are

$$D(\rho_x) = \begin{bmatrix} 1 & & \\ & 1 & -\epsilon \\ & \epsilon & 1 \end{bmatrix} , \quad D(\rho_z) = \begin{bmatrix} 1 & -\epsilon & \\ \epsilon & 1 & \\ & & 1 \end{bmatrix} , \tag{8.40}$$

which gives

$$\varepsilon(k)D(\rho_x) = \begin{bmatrix} a & b & c \\ & A & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\epsilon \\ 0 & \epsilon & 1 \end{bmatrix} = \begin{bmatrix} a & b + \epsilon c & -\epsilon b + c \\ & AD(\rho_x) & \end{bmatrix} = \begin{bmatrix} a & b & c \\ & D(\rho_x)A & \end{bmatrix} , \tag{8.41}$$

we conclude that $b = c = 0$. If we choose $\rho = \rho_z$, we can see $a = 0$, so the effective part is A. This tell us that

$$\begin{pmatrix} m & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & A & \end{pmatrix} = 0 \tag{8.42}$$

. The more general formula is

$$k_\mu \varepsilon^\mu{}_\sigma(k) = 0 . \tag{8.43}$$

Here the proof is not strict and general, though the conclusion is correct. The left relation is $AD(\rho) = AD(\rho)$ for all ρ in $SO(3)$. Based on Schur's lemma, the solution for A must be proportional to the identity.

8.8.4 What is ε mathematically

The equation satisfied by $\varepsilon^{\mu}{}_{\sigma}(k)$ is

$$\sum_{\sigma} D^{*}_{\sigma'\sigma}(W(\Lambda,p))\varepsilon^{\mu}{}_{\sigma}(p) = \sum_{\nu} \mathscr{D}(\Lambda^{-1})^{\mu}{}_{\nu}\varepsilon^{\nu}{}_{\sigma'}(\Lambda p)\,, \quad \forall \rho \in H^{k}_{m}\,. \tag{8.44}$$

Consider

- group G representation (V, ρ_G) such that $\rho_G : G \to \mathrm{Aut}(V)$
- subgroup $H \subset G$ with representation (W, ρ_H)
- a map $\phi : W \to V$, called the equivalent map (or an intertwiner) if for $h \in H$, the map commutes:

$$\begin{array}{ccc} W & \xrightarrow{\phi} & V \\ {\scriptstyle \rho_H(h)}\big\downarrow & & \big\downarrow{\scriptstyle \rho_G|_H(h)} \\ W & \xrightarrow{\phi} & V \end{array}\ \text{, i.e.}\quad \rho_G|_H \circ \phi = \phi \circ \rho_H\,. \tag{8.45}$$

In fact ε plays the role of ϕ, which relates the spin space and the spacetime coordinate space. Explicitly, let $\{v_i\}$ be a basis for V. $\{w_\alpha\}$ is a basis for W. The equivalent equation can be written as

$$\begin{aligned} [\rho_G(h)\circ\phi]w_\alpha = \rho_G(h)\sum_i \phi_{i\alpha}v_i = \sum_i \phi_{i\alpha}\rho_G(h)v_i = \sum_{i,j}\phi_{i\alpha}[\rho_G(h)]_{ji}v_j \\ = [\phi\circ\rho_H(h)]w_\alpha = \phi\sum_\beta[\rho_H(h)]_{\beta\alpha}w_\beta = \sum_{\beta,j}[\rho_H(h)]_{\beta\alpha}\phi_{j\beta}v_j \\ \Rightarrow \quad \sum_i \rho_G(h)_{ji}\phi_{i\alpha} = \sum_\beta \phi_{j\beta}\rho_H(h)_{\beta\alpha}\,. \end{aligned} \tag{8.46}$$

Hence when does ε exist is equivalent to ask when D and $\mathscr{D}|_{H^k_m}$ be intertwined. By representation theory, let (W, ρ_W) be an irrep of H, decompose V into irrep of H

$$V = V_1 \oplus \ldots \oplus V_n\,, \quad \rho^h_i : V_i \to V_i\,, \forall h \in H\,. \tag{8.47}$$

For a non-trivial intertwiner $\phi : W \to V$ to exist, ρ_W must occur among the ρ_i, say $\rho_i \cong \rho_w$, $V_i \cong W$.

Example 8.2. *G is reducible 4-vector reps of $SO^{\uparrow}(1,3)$, $V = \mathbb{R}^4$, as*

$$V = \mathbb{R}^4 = \mathbb{R}\oplus\mathbb{R}^3\,, \tag{8.48}$$

where the 1 dimensional space is the time component invariant under the group $SO(3)$, $\mathbb{R}^3$ is the spatial component that transforms in spin-1 rep of $SO(3)$. We conclude that it can package spin 1 fields using massive vector fields and also the spin 0 fields.

8.9 Massless vector bosons and gauge invariant

8.9.1 The little group at $m = 0$

Convenient choice of k for $k^2 = 0$ to define the little group is $k = (E, 0, 0, E)$,

$$H_0^k = \{\Lambda \in SO^{\uparrow}(1,3) | \Lambda^{\mu}{}_{\nu} k^{\nu} = k^{\mu}\} = ISO(2) \,, \tag{8.49}$$

which is the group of isometries of the Euclidean plane $ISO(2) = \mathbb{R}^2 \rtimes SO(2)$. This group contains two Abelian subgroup $\mathbb{R}^2$, corresponding to the translation $S(\alpha, \beta) \in \mathbb{R}^2$, and rotation $R(\theta) \in SO(2)$. The rotation $R \in SO(2)$ around z-axis can be represented as

$$\mathscr{D}(R)^{\mu}{}_{\nu}(\theta) = \begin{bmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix} \tag{8.50}$$

. The translation can be represented as

$$\mathscr{D}(S)^{\mu}{}_{\nu}(\alpha, \beta) = \begin{bmatrix} 1+\xi & \alpha & \beta & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \xi & \alpha & \beta & 1-\xi \end{bmatrix} \,, \quad \xi = \frac{\alpha^2 + \beta^2}{2} \,. \tag{8.51}$$

Remark 8.7.

$$R(\theta) S(\alpha, \beta) R^{-1}(\theta) = S\left(\alpha\cos\theta + \beta\sin\theta, -\alpha\sin\theta + \beta\cos\theta\right) \,, \tag{8.52}$$

i.e. $\mathbb{R}^2$ *is a normal subgroup of* $ISO(2)$.

We can study the reps of $\mathbb{R}^2$ and $SO(2)$ separately and then lift them to $ISO(2)$. The fact that (follows from Remark 8.7) any non-trivial representation of $\mathbb{R}^2$ lift to an infinite dimensional representation of $ISO(2)$. Such representations would imply a quantum number corresponding to an infinite number of degrees of freedom in addition to momentum p, which is not observed. Hence, we restrict ourselves to the representations of $ISO(2)$ which restrict to trivial reps of $\mathbb{R}^2$. Finite dimensional irreps of $SO(2)$ is one dimensional since $SO(2)$ is Abelian, labelled by $\sigma \in \mathbb{Z}$, which gives

$$D(R(\theta)) |\psi_{k,\sigma}\rangle = e^{i\sigma\theta} |\psi_{k,\sigma}\rangle \,, \tag{8.53}$$

where we will show later that for double cover of $SO(2)$, $\sigma \in \frac{1}{2}\mathbb{Z}$. Overall

$$\begin{aligned} D(W(\theta, \alpha, \beta)) |\psi_{k,\sigma}\rangle &= D(S(\alpha, \beta) \circ R(\theta)) |\psi_{k,\sigma}\rangle \\ = D(S(\alpha, \beta)) D(R(\theta)) |\psi_{k,\sigma}\rangle &= D(S(\alpha, \beta)) e^{i\sigma\theta} |\psi_{k,\sigma}\rangle = e^{i\sigma\theta} |\psi_{k,\sigma}\rangle \,, \end{aligned} \tag{8.54}$$

where the have already used the conclusion that the representation of $S(\alpha, \beta)$ is trivial. Defining $|\psi_{p,\sigma}\rangle =$

$U(L(p))\,|\psi_{k,\sigma}\rangle$, we can see that

$$
\begin{aligned}
U(\Lambda)\,|\psi_{p,\sigma}\rangle = U(\Lambda)U(L(p))\,|\psi_{k,\sigma}\rangle &= U(L(\Lambda p))U(L^{-1}(\Lambda p))U(\Lambda L(p))\,|\psi_{k,\sigma}\rangle \\
&= U(L(\Lambda p))D\left(W(\theta(\Lambda,p),\alpha(\Lambda,p),\beta(\Lambda,p))\right)|\psi_{k,\sigma}\rangle \\
&= U(L(\Lambda p))e^{i\sigma\theta(\Lambda,p)}\,|\psi_{k,\sigma}\rangle = e^{i\sigma\theta(\Lambda,p)}\,|\psi_{\Lambda p,\sigma}\rangle \ .
\end{aligned}
\tag{8.55}
$$

The quantum number σ, specifying $SO(2)$ irrep, is called the helicity. Unlike the 2-component of spin, it is Lorentz invariant. It is similar to spin projected onto direction of motion, but Lorentz invariant. While the irreps of $SO(2)$ are one dimensional, parity maps $\sigma \to -\sigma$. For interaction that preserves parity, such as QED, we consider $|\psi_{p,\pm\sigma}\rangle$ as describing the same particles.

Example 8.3. *Photon has* $\sigma = \pm 1$.

8.9.2 Transformation of $a^\dagger_{p,\sigma}, a_{p,\sigma}$

Following similar procedure as in the section (8.6), we have

$$
\begin{aligned}
U(\Lambda)a^\dagger_{p,\sigma}U^{-1}(\Lambda) &= \frac{\sqrt{2\omega_{\Lambda p}}}{\sqrt{2\omega_p}}e^{i\sigma\theta(\Lambda,p)}a^\dagger_{\Lambda p,\sigma}\,, \\
U(\Lambda)a_{p,\sigma}U^{-1}(\Lambda) &= \frac{\sqrt{2\omega_{\Lambda p}}}{\sqrt{2\omega_p}}e^{-i\sigma\theta(\Lambda,p)}a_{\Lambda p,\sigma}\,.
\end{aligned}
\tag{8.56}
$$

8.9.3 Construction of massless quantum field

Massless scalar field

$\sigma = 0$, via the same calculation as in the massive case, the Lorentz transformation of the field is

$$
U(\Lambda)\phi(x)U^{-1}(\Lambda) = \phi(\Lambda x)\,. \tag{8.57}
$$

Massless vector field

We expect this to describe particles of helicity $\sigma = \pm 1$. As we constructed before,

$$
A^\mu = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{\sigma=\pm1}\left[\varepsilon^\mu{}_\sigma(p)a_{p,\sigma}e^{-ipx} + \varepsilon^\mu{}_\sigma{}^* a^\dagger_{p,\sigma}e^{ipx}\right]\,, \tag{8.58}
$$

we need to find ε to let

$$
U(\Lambda)A^\mu(x)U^{-1}(\Lambda) = \mathscr{D}(\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x)\,. \tag{8.59}
$$

Consider decomposition of 4-vector rep. into $SO(2)$ irrep:

$$
\mathbb{C}^4 = \mathbb{C}_0 \oplus \mathbb{C}_{+1} \oplus \mathbb{C}_{-1} \oplus \mathbb{C}_0\,, \tag{8.60}
$$

where $\mathbb{C}^4$ is the 4-vector rep. of $SO^\uparrow(1,3)$, $\mathbb{C}_0$ is the trivial rep. , $\mathbb{C}_{+1}$ is the $\sigma = +1$ irrep, $\mathbb{C}_{-1}$ is the $\sigma = -1$ irrep. and $\mathbb{C}_0$ is the trivial rep. again. We need to lift these representation to the representation of $ISO(2)$ on which $\mathbb{R}^2$ acts trivially. Explicitly, for

$$\mathbb{C}^4 = \mathbb{C}_0 \oplus \mathbb{C}_{+1} \oplus \mathbb{C}_{-1} \oplus \mathbb{C}_0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle . \tag{8.61}$$

A simple choice of the polarization tensor is

$$\Rightarrow \varepsilon^{\mu}{}_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \tag{8.62}$$

such that the diagram

$$\begin{tikzcd} W \arrow[r, "\phi"] \arrow[d, "\rho_H(h)"'] & V \arrow[d, "\rho_G|_H(h)"] \\ W \arrow[r, "\phi"] & V \end{tikzcd} , \tag{8.63}$$

commutes for $H = SO(2)$. Explicitly,

$$\begin{tikzcd} w_{\pm} \arrow[r] \arrow[d, "\rho_H(\theta)"] & \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \arrow[d, "\rho_G(\theta)"] \\ e^{\pm i\theta} w_{\pm} \arrow[r] & e^{\pm i\theta} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \end{tikzcd} . \tag{8.64}$$

All available choice has been made. Consider $\mathbb{R}^2 \triangleleft ISO(2) = \mathbb{R}^2 \rtimes SO(2)$, the representation restricts on $\mathbb{R}^2$ is trivial representation, however, it is not trivial when lift to the group $ISO(2)$,

$$
\begin{tikzcd}
w_\pm \arrow[r] \arrow[d, "\rho_H(S(\alpha,\beta))"] & & \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \arrow[d, "\rho_G(\theta)"] \\
w_\pm \arrow[r] & \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \neq & \begin{pmatrix} 1+\xi & \alpha & \beta & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \xi & \alpha & \beta & 1-\xi \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \pm i\beta \\ 1 \\ \pm i \\ \alpha \pm \beta \end{pmatrix}
\end{tikzcd}
. \tag{8.65}
$$

This seems tell us that we cannot construct a massless vector filed out of helicity ± 1 creation/annihilation operators. Where is the wrong? This is a very important and interesting question. Let's choose

$$
\varepsilon_\pm(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} , \tag{8.66}
$$

$$
\begin{aligned}
D(R(\theta))\epsilon_\pm(k) &= e^{\pm i\theta} \varepsilon_\pm(k) \\
D(S(\alpha,\beta))\varepsilon_\pm(k) &= \varepsilon_\pm(k) + \frac{1}{\sqrt{2}}(\alpha \pm i\beta)\frac{1}{E} \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix} ,
\end{aligned} \tag{8.67}
$$

and define ${\varepsilon^\mu}_\sigma(p) = {L(p)^\mu}_\nu {\varepsilon^\nu}_\sigma(k)$, where we drop the $\mathscr{D}$ to keep the equation clean.

$$
\begin{aligned}
&U(\Lambda)A^\mu(x)U^{-1}(\Lambda) = \\
&\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \frac{\sqrt{2\omega_{\Lambda p}}}{\sqrt{2\omega_p}} \sum_{\sigma=\pm 1} \left[{\varepsilon^\mu}_\sigma(p) e^{-i\sigma\theta(\Lambda,p)} a_{\Lambda p,\sigma} e^{-ipx} + {\varepsilon^\mu}_\sigma{}^*(p) e^{i\sigma\theta(\Lambda,p)} a^\dagger_{\Lambda p,\sigma} e^{ipx} \right] \\
&\overset{\tilde{p}=\Lambda p}{=} \int \frac{d^3\Lambda^{-1}\tilde{p}}{(2\pi)^3} \frac{\sqrt{2\omega_{\tilde{p}}}}{2\omega_{\Lambda^{-1}\tilde{p}}} \sum_{\sigma=\pm 1} \left[{\varepsilon^\mu}_\sigma(\Lambda^{-1}\tilde{p}) e^{i\sigma\theta(\Lambda,\Lambda^{-1}\tilde{p})} a_{\tilde{p},\sigma} e^{-i\Lambda^{-1}\tilde{p}x} + {\varepsilon^\mu}_\sigma{}^*(\Lambda^{-1}\tilde{p}) e^{i\sigma\theta(\Lambda,\Lambda^{-1}\tilde{p})} a^\dagger_{\tilde{p},\sigma} e^{i\Lambda^{-1}\tilde{p}x} \right] .
\end{aligned} \tag{8.68}
$$

We need to evaluate $\varepsilon(\Lambda^{-1}p)$,

$$\begin{aligned}\varepsilon_\sigma(\Lambda^{-1}p) &= L(\Lambda^{-1}p)\varepsilon_\sigma(k) = \Lambda^{-1}L(p)L(p)^{-1}\Lambda L(\Lambda^{-1}p)\varepsilon_\sigma(k)\\ &= \Lambda^{-1}L(p)W(\Lambda,\Lambda^{-1}p)\varepsilon_\sigma(k)\\ &= \Lambda^{-1}L(p)S(\alpha(\Lambda,\Lambda^{-1}p),\beta(\Lambda,\Lambda^{-1}p))R(\theta(\Lambda,\Lambda^{-1}p))\varepsilon_\sigma(k)\\ &= \Lambda^{-1}L(p)e^{i\sigma\theta(\Lambda,\Lambda^{-1}p)}\left[\varepsilon_\sigma(k)+\frac{1}{\sqrt{2}E}(\alpha+i\sigma\beta)_{(\Lambda,\Lambda^{-1}p)}k\right]\\ &= \Lambda^{-1}e^{i\sigma\theta(\Lambda,\Lambda^{-1}p)}\left[\varepsilon_\sigma(p)+\frac{1}{\sqrt{2}E}(\alpha+i\sigma\beta)_{(\Lambda,\Lambda^{-1}p)}p\right].\end{aligned} \tag{8.69}$$

Substitute it into the equation (8.68), the expression becomes

$$\begin{aligned}(\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x) + \frac{1}{2E}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\,(\Lambda^{-1}p)^\mu\sum_{\sigma=\pm1}&\Big[(\alpha+i\sigma\beta)_{(\Lambda,\Lambda^{-1}p)}a_{p,\sigma}e^{-i(\Lambda^{-1}p,x)}\\ &+(\alpha-i\sigma\beta)a^\dagger_{p,\sigma}e^{i(\Lambda^{-1}p,x)}\Big]\\ &= (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x)+\partial^\mu\Omega(x,\Lambda)\,,\end{aligned} \tag{8.70}$$

The gauge transformation appears naturally! Magic! If a Lagrangian $\mathscr{A}()$ with an action is invariant under

1. $A(x)\to\Lambda^{-1}A(\Lambda x)$,

2. $A(x)\to A(x)+\partial\omega$,

such a Lagrangian will lead to a Lorentz invariant theory. The quantum field we have defined satisfies the following quantities

$$\begin{aligned}\Box A^\mu(x) &= \Box\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{\sigma=-1}^{+1}\left(\varepsilon^\mu{}_\sigma(p)a_{p,\sigma}e^{-ipx}+\varepsilon^\mu{}_\sigma{}^*(p)a^\dagger_{p,\sigma}e^{ipx}\right)\\ &= \int\frac{d^3p}{(2\pi)^3}p^2\frac{1}{\sqrt{2\omega_p}}\sum_{\sigma=-1}^{+1}\left(\varepsilon^\mu{}_\sigma(p)a_{p,\sigma}e^{-ipx}+\varepsilon^\mu{}_\sigma{}^*(p)a^\dagger_{p,\sigma}e^{ipx}\right)=0\,,\end{aligned} \tag{8.71}$$

where we used the massless on-shell condition. For the polarization tensor $\varepsilon_\pm(k)=\frac{1}{\sqrt{2}}(0,1,\pm i,0)^T$, $\varepsilon_\pm(p)=L(p)\varepsilon_\pm(k)$, where $L(p)$ can be chosen as the combination of spatial rotation and boost, $L(p)=R(\vec{p})B(p)$, where $R(\vec{p})$ is the rotation of z-axis into $\vec{p}$ direction and $B(p)$ is the boost along the z-direction, such that

$$B(p)\begin{pmatrix}E\\0\\0\\E\end{pmatrix}=\begin{pmatrix}p^0\\0\\0\\p^0\end{pmatrix}\,, \tag{8.72}$$

this gives $\varepsilon_\pm(p)=R(\vec{p})\varepsilon_\pm(k)$ at t and z component of $\varepsilon_\pm(k)$ vanishes. The rotation of z-axis doesn't change the time component, then we have

$$\Rightarrow\quad \varepsilon^0_\pm(p)=\varepsilon^0_\pm(k)=0\,. \tag{8.73}$$

Notice that we derive $k_\mu \varepsilon^\mu{}_\sigma(k) = 0$ before, now we have

$$\sum_i \hat{k}^i \varepsilon^i_\pm(k) = 0 \quad \Rightarrow \quad \sum_{i,j,l} R(\vec{p})^{ij} \hat{k}^j R(\vec{p})^{il} \varepsilon^l_\pm(k) = \sum_i \hat{p}^i \varepsilon^i_\pm(p)\,, \tag{8.74}$$

where $\hat{k}^i$ is the i-th component of the unit vector and we used the orthogonality of the rotation matrix. In conclusion, we have the two following conditions,

$$\begin{aligned} A^0(x) &= 0\,, \\ \nabla \cdot \vec{A} &= 0\,. \end{aligned} \tag{8.75}$$

Chapter 9

Lagrangian involving a massless helicity 1 particles

9.1 The photon without source

We use A^μ to construct a Lorentz invariant theory. It is equivalent to say that we need to write down a Lagrangian invariant under the transformation $A^\mu \to A^\mu + \partial^\mu \omega$. We further require that there exists a gauge choice such that A^μ satisfies the property above. A natural guess is

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \,, \tag{9.1}$$

which is called the Maxwell Lagrangian. The gauge invariance is obvious since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\Rightarrow \quad \partial_\mu (A_\nu + \partial_\nu \omega) - \partial_\nu (A_\mu + \partial_\mu \omega) = F_{\mu\nu} \,. \tag{9.2}$$

9.1.1 EOM

The Euler-Lagrange equation with respect to A^μ gives

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial \mu A_\nu)} - \frac{\partial \mathscr{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} = -\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = -\Box A^\nu + \partial^\nu \partial_\mu A^\mu \,. \tag{9.3}$$

9.1.2 Gauge theory

Given a solution $A_\mu(x)$ of the EoM (9.3), $A_\mu(x) + \partial_\mu \omega(x)$ also solves the EoM. The physical interpretation is that $A_\mu(x)$ and $A_\mu(x) + \partial_\mu \omega(x)$ are locally equivalent, but it should not be confused with a global symmetry. We can choose $\omega(x)$ (i.e. representation of gauge invariance class conveniently). This is called the gauge fixing.

Coulomb gauge $\nabla \cdot \vec{A} = 0$

Claim: given an A^μ, we can choose a gauge such that $\nabla \cdot \vec{A} = 0$.
Proof: Let $\tilde{A}$ be the field after gauge transformation $\tilde{A}^\mu = A^\mu + \partial^\mu \omega$

$$\partial_i \tilde{A}^i = \left(\partial_i A^i + \partial^i \omega\right) = 0 \,, \quad \Rightarrow \quad -\Delta \omega = -\partial_i A^i \,, \tag{9.4}$$

which is the Poisson equation and it can be solved for reasonable A^i.

Remark 9.1. *Imposing $\nabla \cdot \vec{A}$ does not fix gauge freedom completely. $\nabla \cdot \vec{A}$ preserved by ω which satisfies $\Delta\omega = 0$. This is called the residue gauge degrees of freedom.*

The EoM in Coulomb gauge is

$$\begin{aligned} 0 &= \Box A^0 - \partial^0 \left(\partial_0 + \partial_i A^i\right) = \partial_0 \partial^0 A^0 + \partial_i \partial^i A^0 - \partial^0 \partial_0 A^0 + \partial^0 \partial_i A^i \\ &= \partial_i \partial^i A^0 = -\Delta A^0 \,. \\ 0 &= \Box A^i - \partial^i \partial_\nu \partial A^\nu \\ &= \Box A^i - \partial^i \partial_0 A^0 \,. \end{aligned} \tag{9.5}$$

We can use the residual gauge symmetry to simplify further. Suppose $\tilde{A}^0 = A^0 + \partial^0 \omega = 0$, $\partial^0 \omega = -A^0$,

$$\omega = -\int A^0 dt \,, \quad \Delta\omega = -\int \Delta A^0 = 0 \,, \tag{9.6}$$

where the last equal holds because of the EoM. It is compatible with the Coulomb constraint. The EoM becomes $A^0 = 0$ and $\Box A^i = 0$. These are the equations satisfied by quantum field.

Lorenz gauge

The Lorenz gauge, or the covariant gauge, is

$$\partial_\mu A^\mu = 0 \,. \tag{9.7}$$

Given a solution A^μ, $\partial_\mu \tilde{A} = \partial_\mu \left(A^\mu + \partial^\mu \omega\right) = 0$, the condition is $\Box\omega = -\partial_\mu A^\mu$. In Lorenz gauge $\Box A^\mu = 0$.

9.2 Coupling photons to sources

9.2.1 The gauge covariant derivative

A naive proposal for the coupling (or the interaction) part is $A_\mu f(\phi)$. However, it is not Lorentz invariant. Another guess is $A_\mu f(\phi) \partial^\mu \phi$, however, it is not gauge invariant. The Strategy to fix this problem is to let ϕ transform as well. A natural choice is $\phi \mapsto e^{-i\omega(x)}\phi$:

- by reality of Lagrangian, all terms not involving A^μ or $\partial^\mu \phi$ will be invariant under this transformation.
- $\partial_\mu \phi \mapsto \partial_\mu (e^{-i\omega}\phi) = -i\omega e^{-i\omega}\phi + e^{-i\omega}\partial_\mu \phi$ is the right form to cancel the gauge variation of A^μ

$$\begin{aligned} (\partial_\mu + iA_\mu)\phi \quad &\mapsto \quad \left[\partial_\mu + i(A_\mu + \partial_\mu \omega)\right](e^{-i\omega}\phi) \\ &= e^{-i\omega}(\partial_\mu + iA_\mu)\phi + e^{-i\omega}\cancel{(i\partial_\mu \omega - i\partial_\mu \omega)}\phi \end{aligned} \tag{9.8}$$

.

We can define the gauge covariant derivative $D_\mu \phi = (\partial_\mu + iA_\mu)\phi$, which transforms in the same way as ϕ under the gauge transformation. Coupling ϕ to A^μ by replacing all $\partial_\mu \phi$ by $D_\mu \phi$ yields a gauge invariant Lagrangian,

$$\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \to D_\mu \phi D^\mu \phi - \frac{1}{2} m^2 \phi^2 \,, \tag{9.9}$$

which is called the minimal coupling.

9.2.2 The notion of electromagnetic charge

If there are N different species of fields, denoted by ϕ_n, and we slightly generalize the gauge transformation,

$$\phi_n(x) \to e^{iQ_n\omega(x)}\phi(x)\,, \tag{9.10}$$

where Q_n is the charge of the scalar field under A^μ. The covariant derivative changes

$$D_\mu\phi_n(x) = \left(\partial_\mu - iQ_nA_\mu\right)\phi_n(x)\,. \tag{9.11}$$

We introduce electric charge into the formalism by considering

$$\begin{aligned} A_\mu &\to A_\mu + \frac{1}{e}\partial_\mu\omega \\ \Rightarrow \quad D_\mu\phi(x) &= (\partial_\mu - ieQ_nA_\mu)\phi(x)\,, \end{aligned} \tag{9.12}$$

then Q_n keeps track of the interacted strength.

9.2.3 Mathematical view

Two parts:

- Dropping the spacetime dependence of ω. The gauge transformations forms a group G. The charged fields will be in a representation of a group G.

  Example 9.1. *For electromagnetism, the gauge group is $G = U(1) = \{z \in \mathbb{C}||z| = 1\}$. The electromagnetic field is the one dimensional irreps which are indexed by $n \in \mathbb{Z}$, $e^{i\phi} \mapsto e^{in\phi} \in GL(1,\mathbb{C})$.*

- Spacetime dependence of ω. How to take derivative of $\phi(x)$ when its spacetime variations are partially gauge? We need to distinguish between variation in spacetime vs. in gauge group direction. And we need to choose a suitable connection of A^μ.

9.2.4 The Lagrangian of scalar QED

The Lagrangian is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi(D^\mu\phi)^* - m^2\phi\phi^*\,, \tag{9.13}$$

where the gauge covariant derivative is $D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$ with charge -1. The Lagrangian is invariant under the transformation

$$\begin{aligned} A_\mu &\to A_\mu + \frac{1}{e}\partial_\mu\omega \\ \phi &\to e^{-i\omega}\phi \end{aligned} \tag{9.14}$$

9.2.5 Anti-particle

Recall that ϕ and ϕ^* can be treated as independent degrees of freedom.

$$(D_\mu\phi)^* = (\partial_\mu - ieA_\mu)\phi^*\,. \tag{9.15}$$

The opposite sign here guides us to introduce the states of opposite charges. The creation and annihilation operators are

$$\begin{aligned} a_p,\, a_p^\dagger :\ [a_k, a_p^\dagger] &= (2\pi)^3\delta^{(3)}(\vec{k}-\vec{q}) \\ b_p,\, b_p^\dagger :\ [b_k, b_p^\dagger] &= (2\pi)^3\delta^{(3)}(\vec{k}-\vec{q})\,, \end{aligned} \tag{9.16}$$

where $b_p, b_p^\dagger$ are creation and annihilation operator for states of opposite charge and we called they are anti-particles. Different particle type, their creation/annihilation operator commutes: $[a_k^\cdot, b_p^\cdot] = 0$. The fields are

$$\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right) \\ \phi^*(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p^\dagger e^{ipx} + b_p e^{-ipx}\right) \\ \omega_p &= \sqrt{\vec{p}^2+m^2}\,. \end{aligned} \tag{9.17}$$

The particles and antiparticles have the same mass and both have positive energy. Combining quantum mechanics and special relativity implies the existence of anti-particles.

In summary, the route of this chapter is quantizing photons (defining a massless quantum vector field for helicity ± 1 states) $\to$ gauge invariance (with gauge group $U(1)$) $\to$ necessity of complex reps $\to$ the existence of anti-particles.

Chapter 10

Scalar QED

10.1 The photon propagator (Coulomb gauge)

The photon propagator

$$
\begin{aligned}
&\langle 0|\,\mathscr{T}\left\{A^\mu(x)A^\nu(y)\right\}|0\rangle \\
&=\sum_{\sigma=\pm1}\sum_{\sigma'=\pm1}\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\Big[\Theta(x^0-y^0)e^{i(-px+qy)}\langle 0|\,a_{p,\sigma}a^\dagger_{q,\sigma}\,|0\rangle\,\varepsilon^\mu{}_\sigma(p)\varepsilon^\nu{}_{\sigma'}{}^*(p) \\
&+\Theta(x^0-y^0)e^{i(px-qy)}\langle 0|\,a_{q,\sigma'}a^\dagger_{p,\sigma}\,|0\rangle\,\varepsilon^\mu{}_\sigma{}^*\varepsilon^\nu{}_{\sigma'}(q)\Big] \\
&=\sum_{\sigma=\pm1}\sum_{\sigma'=\pm1}\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\Big[\Theta(x^0-y^0)e^{i(-px+qy)}\langle 0|\,[a_{p,\sigma},a^\dagger_{q,\sigma}]\,|0\rangle\,\varepsilon^\mu{}_\sigma(p)\varepsilon^\nu{}_{\sigma'}{}^*(p) \\
&+\Theta(x^0-y^0)e^{i(px-qy)}\langle 0|\,[a_{q,\sigma'},a^\dagger_{p,\sigma}]\,|0\rangle\,\varepsilon^\mu{}_\sigma{}^*\varepsilon^\nu{}_{\sigma'}(q)\Big] \\
&=\sum_{\sigma=\pm1}\sum_{\sigma'=\pm1}\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\Big[\Theta(x^0-y^0)e^{i(-px+qy)}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{\sigma\sigma'}\varepsilon^\mu{}_\sigma(p)\varepsilon^\nu{}_{\sigma'}{}^*(p) \\
&+\Theta(x^0-y^0)e^{i(px-qy)}(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})\delta_{\sigma'\sigma}\varepsilon^\mu{}_\sigma{}^*\varepsilon^\nu{}_{\sigma'}(q)\Big] \\
&=\sum_{\sigma=\pm1}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\Big[\varepsilon^\mu{}_\sigma(p)\varepsilon^\nu{}_\sigma{}^*(p)e^{-ip(x-y)}\Theta(x^0-y^0)+\varepsilon^\nu{}_\sigma{}^*(p)\varepsilon^\nu{}_\sigma(p)e^{ip(x-y)}\Theta(x^0-y^0)\Big]\,.
\end{aligned}
\tag{10.1}
$$

We need to evaluate the summation $\sum_\sigma \varepsilon^\mu{}_\sigma(p)\varepsilon^\nu{}_\sigma(p)$, using the discussion before $\varepsilon_\pm(p)=R(\vec{p})\varepsilon_\pm(k)$, we have

$$\sum_\sigma \varepsilon^\mu{}_\sigma(p)\varepsilon^\nu{}_\sigma(p)=\sum_{\sigma=\pm1}R^\mu{}_\kappa(\vec{p})\varepsilon^\kappa{}_\sigma(k)R^\nu{}_\lambda(\vec{p})\varepsilon^\lambda{}_\sigma{}^*(k)\,, \tag{10.2}$$

where $R(\vec{p})$ is the rotational transformation that rotate $\vec{k}$ into $\vec{p}$. Recall that we have the expression of $\varepsilon_\pm = \frac{1}{\sqrt{2}}(0,1,\pm i,0)^T$, then

$$
\begin{aligned}
\sum_\sigma \varepsilon^\mu{}_\sigma(p)\varepsilon^\nu{}_\sigma(p) &= \frac{1}{2}\sum_{\sigma=\pm1} R^\mu{}_\kappa(\vec{p})R^\nu{}_\lambda(\vec{p}) \left[\begin{pmatrix}0\\1\\\sigma i\\0\end{pmatrix}\begin{pmatrix}0 & 1 & -\sigma i & 0\end{pmatrix}\right]^{\kappa\lambda} \\
&= \frac{1}{2}\sum_{\sigma=\pm1} R^\mu{}_\kappa(\vec{p})R^\nu{}_\lambda(\vec{p}) \begin{pmatrix}0 & & & 0\\ & 1 & -\sigma i & \\ & \sigma i & 1 & \\ 0 & & & 0\end{pmatrix}^{\kappa\lambda} \\
&= R^\mu{}_\kappa(\vec{p})R^\nu{}_\lambda(\vec{p}) \begin{pmatrix}0 & & & 0\\ & 1 & 0 & \\ & 0 & 1 & \\ 0 & & & 0\end{pmatrix}^{\kappa\lambda} \\
&= R^\mu{}_\kappa(\vec{p})R^\nu{}_\lambda(\vec{p}) \left(\begin{array}{c|c} 0 & \\ \hline & \delta^{ij} - \frac{k^i k^j}{|k|^2}\end{array}\right)^{\kappa\lambda} \quad \text{for} \quad \vec{k} = (0,0,E)^T \\
&\overset{R(\vec{p})\hat{\vec{k}}=\vec{p}}{=} \left(\begin{array}{c|c} 0 & \\ \hline & \delta^{ij} - \frac{p^i p^j}{|p|^2}\end{array}\right)^{\mu\nu} = P^{\mu\nu}\,,
\end{aligned}
\tag{10.3}
$$

though it seems that the calculation relies on the choice of k, we claim that it is true for all k. Now the integral becomes

$$
= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}P^{\mu\nu}(p)e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left[e^{-i\omega_p(x^0-y^0)}\Theta(x^0-y^0) + e^{i\omega_p(x^0-y^0)}\Theta(y^0-x^0)\right]\,, \tag{10.4}
$$

where we use the integral property: the $e^{\pm i\vec{p}\cdot(\vec{x}-\vec{y})}$ integration will be same. Noticing that

$$
e^{-i\omega_p(x^0-y^0)}\Theta(x^0-y^0) + e^{i\omega_p(x^0-y^0)}\Theta(y^0-x^0) = -\frac{2\omega_p}{2\pi i}\lim_{\epsilon\to0}\int\frac{d\omega e^{i\omega(x^0-y^0)}}{\omega^2-\omega_p^2+i\epsilon}\,, \tag{10.5}
$$

with $\omega_p^2 = \vec{p}^2 + m^2 = \vec{p}^2$, $\omega^2 - \omega_p^2 = p^2$. Combine the integration, we have

$$
\langle 0|\mathscr{T}\left\{A^\mu(x)A^\nu(y)\right\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{iP^{\mu\nu}(p)}{p^2+i\epsilon}e^{ip(x-y)}\,. \tag{10.6}
$$

We extract the momentum space propagator

$$
\begin{aligned}
\langle 0|\mathscr{T}\left\{A^\mu(x)A^\nu(y)\right\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{iP^{\mu\nu}(p)}{p^2+i\epsilon}e^{ip(x-y)} = i\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\Pi^{\mu\nu}(p)\,, \\
i\Pi^{\mu\nu}(p) = \frac{iP^{\mu\nu}(p)}{p^2+i\epsilon}\,.
\end{aligned}
\tag{10.7}
$$

(Derived via path integral), $\Pi^{\mu\nu}$ in covariant gauges are

$$\Pi^{\mu\nu} = -\frac{g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon} \,, \tag{10.8}$$

where ξ corresponds to a choice of covariant gauge. Because this is a gauge choice, it must be dropped out of the final result. We then have two options in the later calculation,

- keep ξ-dependence in the calculation.

- fix ξ value.

    1. $\xi = 1$, Feynman gauge
    $$i\Pi^{\mu\nu} = -\frac{ig^{\mu\nu}}{p^2 + i\epsilon} \tag{10.9}$$
    .

    2. $\xi = 0$, 't Hooft gauge
    $$i\Pi^{\mu\nu} = -i\frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon} \,. \tag{10.10}$$

10.2 LSZ for scalar QED

10.2.1 complex scalar field

Recall that the proof of LSZ is based on

$$\begin{aligned} i\int d^4x e^{ipx}(\Box + m^2)\phi(x) &= \sqrt{2\omega_p}\left(a_p(\infty) - a_p(-\infty)\right) \,, \\ -i\int d^4x e^{-ipx}(\Box + m^2)\phi(x) &= \sqrt{2\omega_p}\left(b_p^\dagger(\infty) - b_p^\dagger(-\infty)\right) \,, \end{aligned} \tag{10.11}$$

analogous relation for $\phi^*(x)$,

$$\begin{aligned} -i\int d^4x e^{-ipx}(\Box + m^2)\phi^*(x) &= \sqrt{2\omega_p}\left(a_p^\dagger(\infty) - a_p^\dagger(-\infty)\right) \,, \\ i\int d^4x e^{ipx}(\Box + m^2)\phi^*(x) &= \sqrt{2\omega_p}\left(b_p(\infty) - b_p(-\infty)\right) \,, \end{aligned} \tag{10.12}$$

from these expression, we justify

- $\phi(x)$ insertions yield incoming e^+ (anti-particle, modes created by $b_p^\dagger$) and outgoing e^- (particle, modes created by $a_p^\dagger$).

- $\phi^*(x)$ insertions yield incoming e^-, outgoing e^+.

10.2.2 Massless vector field (gauge boson, vector boson, photons)

The LSZ for A^μ is

$$\begin{aligned} i\int d^4x e^{ipx}\Box A^\mu(x) &= \sqrt{2\omega_p}\sum_{\sigma=\pm1}\left[\varepsilon^\mu{}_\sigma(p)(a_{p,\sigma}(\infty)-a_{p,\sigma}(-\infty))\right] \\ -i\int d^4x e^{-ipx}\Box A^\mu(x) &= \sqrt{2\omega_p}\sum_{\sigma=\pm1}\varepsilon^\mu{}_\sigma{}^*(p)\left(a^\dagger_{p,\sigma}(\infty)-a^\dagger_{p,\sigma}(-\infty)\right) . \end{aligned} \tag{10.13}$$

We want to project onto a particular helicity of the external state $a^\dagger_{p,\sigma}\left|0\right\rangle$. Recall that $\varepsilon_\pm = \frac{1}{\sqrt{2}}(0,1,\pm i,0)^T$, we have

$$\begin{aligned} \varepsilon^\mu_\pm(k)\varepsilon^*_{\mu\pm}(k) &= -1\,, \\ \varepsilon^\mu_\pm(k)\varepsilon^*_{\mu\mp}(k) &= 0\,, \\ \varepsilon^\mu{}_\sigma(k)\varepsilon_{\mu\sigma'}{}^*(k) &= -\delta_{\sigma\sigma'}\,, \\ \varepsilon_\pm(p) = R(p)\varepsilon_\pm(k) \quad &\Rightarrow \quad \varepsilon^\mu{}_\sigma(p)\varepsilon_{\mu\sigma'}{}^*(p) = -\delta_{\sigma\sigma'}\,,. \end{aligned} \tag{10.14}$$

Project onto particular helicity,

$$\begin{aligned} i\int d^4x e^{ipx}\Box(-\varepsilon_{\mu,\sigma}{}^*(p))A^\mu(x) &= \sqrt{2\omega_p}\left[a_{p,\sigma}(\infty)-a_{p,\sigma}(-\infty)\right] \\ -i\int d^4x e^{-ipx}\Box(-\varepsilon_{\mu,\sigma}(p))A^\mu(x) &= \sqrt{2\omega_p}\left[a^\dagger_{p,\sigma}(\infty)-a^\dagger_{p,\sigma}(-\infty)\right] . \end{aligned} \tag{10.15}$$

This tells us

- $-\varepsilon_{\mu,\sigma}(p)A^\mu(x)$ insertion yields incoming particle of momentum p and helicity σ.
- $-\varepsilon_{\mu,\sigma}{}^*(p)A^\mu(x)$ insertion yields outgoing particle of momentum p and helicity σ.

In Feynman rules, incoming particle comes with a factor of $-\varepsilon_{\mu,\sigma}(p)$, outgoing with a factor of $-\varepsilon^*_{\mu,\sigma}(p)$ with a contraction over μ (with the μ index of $P^{\mu\nu}$ occurring in the photon propagator).

Remark 10.1. *Signs will cancel in $|\left\langle f\right| S \left|i\right\rangle|^2$, so we can choose $+\varepsilon^\mu{}_\sigma, +\varepsilon^\mu{}_\sigma{}^*$ instead.*

10.3 Ward identity

From our analysis of quantum fields, $\varepsilon^\mu(p) \sim \varepsilon^\mu(p) + c\hat{p}^\mu$. Writing an amplitude involving an external photon as

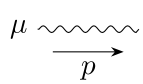
$$\mathscr{M} = \varepsilon_\mu(p)\mathscr{M}^\mu \stackrel{!}{=} \left(\varepsilon^\mu(p) + c\hat{p}^\mu\right)\mathscr{M}_\mu \quad \Rightarrow \quad p_\mu\mathscr{M}^\mu \stackrel{!}{=} 0\,, \tag{10.16}$$

which is the Ward identity. We will consider this identity in the learning of path integral. From now, we will see that this holds in special examples.

10.4 Feynman rules for scalar QED

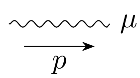
10.4.1 Incoming, outgoing particles

- Incoming photons:



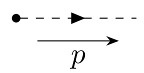
$$= \varepsilon^{\mu}{}_{\sigma}(p)\,. \tag{10.17}$$

- Outgoing photon



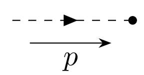
$$= \varepsilon^{\mu}{}_{\sigma}(p)^*\,. \tag{10.18}$$

- Incoming e^-



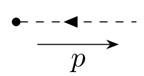
$$\tag{10.19}$$

- Outgoing e^-



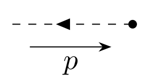
$$\tag{10.20}$$

- Incoming e^+



$$\tag{10.21}$$

- Outgoing e^+



$$\tag{10.22}$$

where all the dots represents the external points.

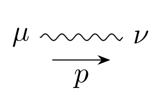
10.4.2 Propagators

Suppose ψ_1 and ψ_2 are two fields and there propagator is

$$\langle 0|\, \mathscr{T}\left\{\psi_1(x)\psi_2(y)\right\}|0\rangle = i\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\Pi(p)\,, \tag{10.23}$$

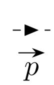
- Photon

$$i\Pi^{\mu\nu} = \frac{-i}{p^2+i\epsilon}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right] = \tag{10.24}$$



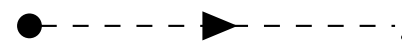
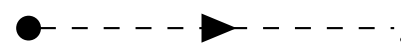
- Scalar field

$$i\Pi_S = \frac{i}{p^2-m^2+i\epsilon} = \phi^*(x) \quad \phi(y)\,. \tag{10.25}$$



- The propagator $\langle 0|\, \mathscr{T}\left\{\phi(x)\phi(y)\right\}|0\rangle = \langle 0|\, \mathscr{T}\left\{\phi^*(x)\phi^*(y)\right\}|0\rangle \equiv 0$ as $[a_p, b_p^\dagger] = [a_p^\dagger, b_p] = 0$.

Hence,

- incoming e^- is ϕ^* insertion .
- incoming e^+ is ϕ insertion .
- outgoing e^- is ϕ insertion .
- outgoing e^+ is ϕ^* insertion .

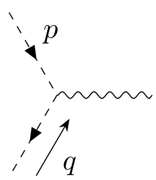
Remark 10.2. *Diagrams come with particle or charge flow arrow.*

10.4.3 Vertices

Expand out covariant derivative, $D_\mu = \partial_\mu + ieA_\mu$, the Lagrangian is

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi(D^\mu\phi)^* - m\phi\phi^* \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\Box + m^2)\phi - ieA_\mu\left(\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi\right) + e^2A_\mu A^\mu\phi\phi^* .\end{aligned} \tag{10.26}$$

Comes from the Lagrangian, the vertex should have the value,

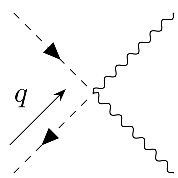


$$= -ie(\pm ip^\mu \pm iq^\mu)i\,, \tag{10.27}$$

where the last i comes from the expansion of $e^{i\int \mathscr{L}_{\text{int}}dt}$ and we will determine the signs later

Remark 10.3. *The physical quantity is $|\mathscr{M}|^2$, so what does matter is the relative signs.*

. Because $A_\mu A^\mu = g_{\mu\nu}A^\mu A^\nu$, we have

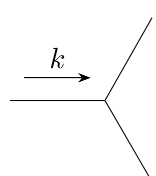


$$= e^2 i2g_{\mu\nu}\ , \tag{10.28}$$

where the i comes from the expansion of the $e^{i\int \mathscr{L}_{\text{int}}d^4x}$ again.

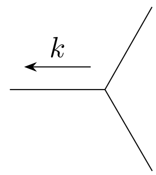
Remark 10.4. *We cannot normalize with a factor of $1/2$, because we cannot change the term (we have gauge invariance requirement).*

Now we need to derive the signs. Such a vertex term comes from the term $A_\mu\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$. Recall that for incoming particle carrying a momentum k, the Feynman diagram is



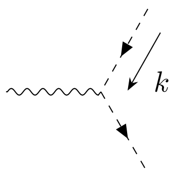
, (10.29)

for the outgoing particle carrying a momentum k,

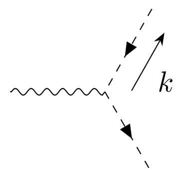


$$. \tag{10.30}$$

So for the term $\phi^* \partial^\mu \phi$, there are two possible diagram

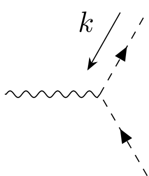


$$\sim \partial^\mu e^{-ikx} = -ik^\mu \,,$$

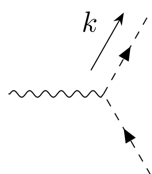


$$\sim \partial^\mu e^{ikx} = ik^\mu \,. \tag{10.31}$$

For the term $-\phi \partial^\mu \phi^*$, also two possible diagram,



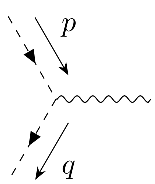
$$\sim -\partial^\mu e^{-ikx} = ik^\mu \,,$$



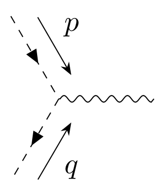
$$\sim -\partial^\mu e^{ikx} = -ik^\mu \,. \tag{10.32}$$

We conclude that the sign will be $-ik^\mu$ if momentum and particle flow arrow are aligned, $+ik^\mu$ if they are anti-aligned.

Example 10.1.



$$= -ie(-p^\mu - q^\mu) \,,$$



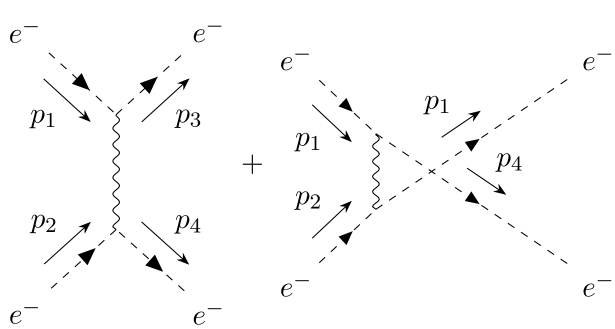
$$= ie(-p^\mu + q^\mu) \,. \tag{10.33}$$

10.5 Møller scattering

The Møller scattering process is

$$e^- e^- \to e^- e^- \,. \tag{10.34}$$

The tree-level Feynman diagram is



$$= i\mathscr{M}_E + i\mathscr{M}_u \,, \tag{10.35}$$

$$
\begin{aligned}
i\mathscr{M}_e &= ie(-p_1-p_3)^\mu \left(-i\frac{g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2}\right) ie(-p_2-p_4)^\nu \\
&= ie^2(p_1+p_3)^\mu(p_2+p_4)_\mu \frac{1}{(p_3-p_1)^2}\,,
\end{aligned}
\tag{10.36}
$$

because $(p_1+p_3)\cdot k = (p_1+p_3)\cdot(p_3-p_1) = p_3^2 - p_1^2 = 0$, the result is indeed ξ-independent.

$$
i\mathscr{M}_u = ie^2\frac{(p_1+p_4)^\mu(p_2+p_3)_\mu}{(p_4-p_1)^2}\,,
\tag{10.37}
$$

the cross section is

$$
\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{1}{64\pi^2 E_{\text{CM}}^2}|\mathscr{M}|^2 = \frac{e^4}{64\pi^2 E_{\text{CM}}^2}\left[\frac{(p_1+p_3)(p_2+p_4)}{(p_3-p_1)^2} + \frac{(p_1+p_4)(p_2+p_3)}{(p_4-p_1)}\right]^2 \\
&= \frac{\alpha^2}{4s}\left[\frac{s-u}{t} + \frac{s-t}{u}\right]^2\,, \quad \alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}\,,
\end{aligned}
\tag{10.38}
$$

where the subscript CM means the center of mass frame.

Chapter 11

Finite dimensional representation of the Lorentz group

11.1 Lie group and Lie algebras

Lie groups are differentiable manifold that carries a group structure. Much of the structure is reviewed by studying them independently in a neighbourhood of the identity, which is called the Lie algebra, a vector space with multiplication.

Example 11.1. *For matrix groups, containing matrices of some dimension, no longer necessary invertible. There exists a map:*

$$\begin{aligned} \exp : \mathscr{G} &\to G \\ A &\mapsto e^{iA} . \end{aligned} \tag{11.1}$$

The group multiplication is encoded in commutator of Lie algebra,

$$[\cdot, \cdot] : \mathscr{G} \times \mathscr{G} \to \mathscr{G} . \tag{11.2}$$

For matrix group, it is natural to have $[A, B] = AB - BA$. Finite dimensional Lie group have finite dimensional Lie algebra.

11.2 Representations of Lie group and Lie algebra

Definition 11.1. *The representation of Lie group on vector space of dimension n is a map:*

$$\rho_G : G \to Aut(V) \cong GL(n) , \tag{11.3}$$

where $GL(n)$ requires choices of the basis of V.

Definition 11.2. *Representation of Lie algebra on same V,*

$$\rho_{\mathscr{G}} ; \mathscr{G} \to End(V) \cong M(n) , \tag{11.4}$$

where $M(n)$ is the matrix collection and it doesn't require the matrix is invertible.

$\rho_G, \rho_{\mathscr{G}}$ both respect the group/algebra structure,

$$\begin{aligned} \rho_G(g_1 g_2) &= \rho_G(g_1) \circ \rho(g_2)\,, \quad \text{for} \quad \forall g_1, g_2 \in G\,, \\ \rho_{\mathscr{G}}(\alpha A + \beta B) &= \alpha \rho_{\mathscr{G}}(A) + \beta \rho_{\mathscr{G}}(B)\,, \quad \forall \alpha, \beta \in \mathbb{R}/\mathbb{C}\,, A, B \in \mathscr{G}\,, \\ \rho_{\mathscr{G}}[A, B] &= [\rho_{\mathscr{G}}(A), \rho_{\mathscr{G}}(B)]\,. \end{aligned} \tag{11.5}$$

Theorem 11.1. *Finite dimensional representations of connected and simply connected Lie group are in one by one relation to finite dimensional representations of their Lie algebras.*

Definition 11.3. *Connectness: A topological space is connected if it cannot be written as the adjoint union of two non-empty open sets.*

Lemma 11.1. *The continuous image of a connected space is connected.*

In fact, equivalent definition of connectness is: X is connected if all continuous functions from X to the space $\{1, -1\}$ with the discrete topology are constant.

Definition 11.4. *Simple-connectness: X is simply connected if all continuous maps $S^1 \to X$ can be continuously contracted to a point.*

11.3 Connected components of the Lorentz group $O(1,3)$

There exists two group homomorphism $\phi_i : O(1,3) \to \{\pm 1\} \quad i = 1, 2$.

$$\begin{aligned} \phi_1 = \phi_{\text{det}} : &\quad A \mapsto \det(A) \\ \phi_2 = \phi_{\text{sgn}} : &\quad A \mapsto \text{sgn}(A^0{}_0)\,. \end{aligned} \tag{11.6}$$

component	in ϕ_{det}	in ϕ_{sgn}
$SO^{\uparrow}(1,3)$	1	1
$\mathbb{P}SO^{\uparrow}(1,3)$	-1	1
$\mathbb{T}SO^{\uparrow}(1,3)$	-1	-1
$\mathbb{T}\mathbb{P}SO^{\uparrow}(1,3)$	1	-1

$SO^{\uparrow}(1,3)$ is connected (in fact, it is path connected). $O(1,3)$ has 4 connected component.

11.4 The simply connected cover $SL(2,\mathbb{C})$ of $SO^{\uparrow}(1,3)$

$SO^{\uparrow}(1,3)$ is not simply connected. since the subgroup $SO(3)$ is not simply connected. Its double cover $SL(2,\mathbb{C})$ (2×2 matrices with unit determinant) is simply connected. We construct the group homomorphism: $K : SL(2,\mathbb{C}) \overset{2:1}{\to} SO^{\uparrow}(1,3)$. Let $H = \{A \in M(2,\mathbb{C}) | A^{\dagger} = A\} = \langle \sigma_0, \vec{\sigma} \rangle$. Consider

$$\begin{aligned} \phi : \mathbb{R}^4 &\to H \\ x &\mapsto x^{\mu} \sigma_{\mu} \end{aligned} \tag{11.7}$$

where the determinant

$$\det(x^{\mu}\sigma_{\mu}) = \begin{vmatrix} x^0 + x^3 & x^1 - ix^3 \\ x^1 + ix^2 & x^0 - x^3 \end{vmatrix} = x^{\mu} x_{\mu}\,. \tag{11.8}$$

Let $SL(2,\mathbb{C})$ act on H by conjugation,

$$\begin{aligned} G \in SL(2,\mathbb{C}) : H &\to H \\ A &\mapsto GAG^\dagger\,, \end{aligned} \tag{11.9}$$

Definition 11.5. *We could then define the group homomorphism $K : SL(2,\mathbb{C}) \to SO^\uparrow(1,3)$, $K(G) : \mathbb{R}^4 \to \mathbb{R}^4$ such that*

$$G\phi(x)G^\dagger = (K(G)x)^\mu \sigma_\mu\,. \tag{11.10}$$

We claim that $K(G) \in O^\uparrow(1,3)$, $K(G)$ is linear follows from the linearity of ϕ. $K(G)$ preserves Minkowski norm, since

$$[K(G)x]_\mu [K(G)x]^\mu = \det\left([K(G)x]^\mu \sigma_\mu\right) = \det\Big(G\phi G^\dagger\Big) = \det(G)\det(\phi)\det\Big(G^\dagger\Big) = \det\phi = x_\mu x^\mu\,. \tag{11.11}$$

As continuous image of connected space is connected, we conclude that $K(G) \in SO^\uparrow(1,3)$. We indeed construct a group homomorphism from $SL(2,\mathbb{C})$ to $SO^\uparrow(1,3)$.

Remark 11.1. *The kernel of the map K is*

$$\operatorname{Ker} K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}\,, \tag{11.12}$$

thus the map K is two to one.

11.5 The Lie algebra $SO(1,3)$

Example 11.2. *The transformation of $SO(1,3)$ takes the following form, for example, the rotation around x-axis and the boost along the x axis is*

$$\begin{aligned} &\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_x & \sin\theta_x \\ & & -\sin\theta_x & \cos\theta_x \end{pmatrix} \overset{infinitesimal}{\mapsto} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \theta_x \\ & & -\theta_x & 1 \end{pmatrix} \\ &\begin{pmatrix} \cosh\beta_x & \sinh\beta_x & & \\ \sin\beta_x & \cosh\beta_x & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \overset{infinitesimal}{\mapsto} \begin{pmatrix} 1 & \beta_x & & \\ \beta_x & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\,. \end{aligned} \tag{11.13}$$

All matrices on the right have the form

$$\mathbb{1} + i\theta_i J_i\,, \quad \mathbb{1} + i\beta_i K_i\,. \tag{11.14}$$

The $SO(1,3)$ is generated by $\{J_1, J_2, J_3.K_1, K_2, K_3\}$ with the following commutation relation

$$\begin{aligned}[J_i, J_j] &= i\varepsilon_{ijk} J_k\,, \\ [J_i, K_j] &= i\varepsilon_{ijk} K_k\,, \\ [K_i, K_j] &= -i\varepsilon_{ijk} J_k\,.\end{aligned} \tag{11.15}$$

Or more compactly, define

$$V^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}\,, \tag{11.16}$$

with the commutator

$$[V^{\mu\nu}, V^{\rho\sigma}] = i\left(g^{\nu\rho}V^{\mu\sigma} + g^{\mu\sigma}V^{\nu\rho} - g^{\mu\rho}V^{\nu\sigma} - g^{\nu\sigma}V^{\mu\rho}\right).E \tag{11.17}$$

We introduce the generator

$$J_i^+ = \frac{1}{2}(J_i + iK_i)\,, \quad J_i^- = \frac{1}{2}(J_i - iK_i)\,, \tag{11.18}$$

with the commutator

$$\begin{aligned}[J_i^+, J_j^+] &= i\varepsilon_{ijk} J_k^+\,, \\ [J_i^-, J_j^-] &= i\varepsilon_{ijk} J_k^-\,, \\ [J_i^+, J_j^-] &= 0\,,\end{aligned} \tag{11.19}$$

based on these generators, we conclude that $SO(1,3) \cong SU(2) \oplus SU(2)$. The irreps of $SO(1,3)$ (and thus $SL(2,\mathbb{C})$) are labelled via the tensor product of two irreps of $SU(2)$, which means that they are labelled by two positive half-integers (A, B), while the irreps have dimension $(2A+1)(2B+1)$. The representation (A, B) of $SU(2) \oplus SU(2)$ (or of $SL(2,\mathbb{C})$) acts on the tensor product of the $SU(2)$ representation spaces V_A, V_B as

$$\begin{aligned}\rho^{so(1,3)}(A,B)(\alpha^+ + \alpha^-)(V \otimes W) = \left(\rho_A^{su(2)}(\alpha^+)V \otimes W + V \otimes \rho_B^{su(2)}(\alpha^-)W\right)\,, \\ \forall V \otimes W \in V_A \otimes V_B\,.\end{aligned} \tag{11.20}$$

To help identify (A, B) with irreps are already known, it helps to strict to rotations. Since

$$\begin{aligned} so(3) \cong su(2) &\hookrightarrow su(2) \oplus su(2) \\ \theta_i J_i &\mapsto \theta_i (J_i^+ + J_i^-)\,,\end{aligned} \tag{11.21}$$

$\rho_{(A,B)}$ restricts to the tensor product representation $\rho_A \otimes \rho_B$ of $SU(2)$.

irreps of $so(1,3)$	$(0,0)$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$
irreps of $su(2)$	$0 \otimes 0 = 0$	$\frac{1}{2} \otimes 0 = \frac{1}{2}$	$0 \otimes \frac{1}{2} = \frac{1}{2}$	$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

irreps of $so(1,3)$	$(1,0)$	$(0,1)$	$(1,1)$
irreps of $su(2)$	$1 \otimes 0 = 1$	$0 \otimes 1 = 1$	$1 \otimes 1 = 0 \oplus 1 \oplus 2$

- $0 \otimes 0 = 0$ is the trivial representation (scalar).

- $\frac{1}{2} \otimes 0$ and $0 \otimes \frac{1}{2}$ are two spinors reps.
- $(\frac{1}{2} \otimes \frac{1}{2})$ are 4-vector reps.
- $1 \otimes 0$ and 0×1 are self dual reps.
- $1 \otimes 1$ is traceless symmetric reps.

Chapter 12

Spinors

12.1 Spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$

Recall the two dimension spin $\frac{1}{2}$ irreps of $SU(2)$. We can choose Pauli matrices as generators of $SU(2)$, with commutator and anti-commutator

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k\,, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\,. \tag{12.1}$$

The 2 dimensional $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ irreps of $so(1,3)$ are spanned by

$$\begin{aligned} &(\frac{1}{2}, 0) \quad \rho_{(\frac{1}{2},0)}(J_i^-) = \frac{\sigma_i}{2} \quad \rho_{(\frac{1}{2},0)}(J_i^+) = 0 \\ &(0, \frac{1}{2}) \quad \rho_{(0,\frac{1}{2})}(J_i^-) = 0 \quad \rho_{(0,\frac{1}{2})}(J_i^+) = \frac{\sigma_i}{2}\,. \end{aligned} \tag{12.2}$$

The generators of rotation and boosts can be represented as

$$\begin{aligned} (\frac{1}{2}, 0) \quad & \rho_{(\frac{1}{2},0)}(J_i) = \rho_{(\frac{1}{2},0)}(J_i^+ + J_i^-) = \frac{\sigma_i}{2} \\ & \rho_{(\frac{1}{2},0)}(K_i) = \rho_{(\frac{1}{2},0)}(i(J_i^+ - J_i^-)) = \frac{i}{2}\sigma_i \\ (0, \frac{1}{2}) \quad & \rho_{(0,\frac{1}{2})}(J_i) = \frac{\sigma_i}{2} \\ & \rho_{(0,\frac{1}{2})}(K_i) = -\frac{i}{2}\sigma_i\,. \end{aligned} \tag{12.3}$$

Remark 12.1. *Such a representation is not a unitary representation since it is finite dimensional. There two representations are related by conjugation. We call these two are conjugated representation with respect to the other one.*

Theorem 12.1. *Let G and H be topological groups, and assume that G is simply connected. Let U be a neighbourhood of the identity of G. Then for any local homomorphism $U \to H$, it can be extended to a homomorphism $G \to H$. For us, $H = Aut(V)$, $\exp|_u : u \xrightarrow{\sim} U$.*

Definition 12.1. *A spinor is an element of the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representation space $V_{(\frac{1}{2},0)}$ or $V_{(0,\frac{1}{2})}$, we define*

- $\psi_R \in V_{(0,\frac{1}{2})}$ *is the right-handed Weyl spinor.*

- $\psi_L \in V_{(\frac{1}{2},0)}$ *is the left-handed Weyl spinor.*

Our convention (A, B) *corresponds to the generator* J_i^- *and* J_i^+ *separately.*

Under a Lorentz transformation, writing $J_i^R = \rho_R(J_i)$ and $K_i^R = \rho_R(K_i)$,

$$\begin{aligned} \psi_R &\to \exp\left(i\theta_i J_i^{(0,\frac{1}{2})} + i\beta_i K_i^{(0,\frac{1}{2})}\right)\psi_R = e^{\frac{1}{2}(i\theta_i\sigma_i + \beta_i\sigma_i)}\psi_R \\ \psi_L &\to \exp\left(\frac{1}{2}(i\theta_i - \beta_i)\sigma_i\right)\psi_L\,. \end{aligned} \tag{12.4}$$

12.2 Lorentz invariance and spinors

To construct Lorentz invariant Lagrangian out of spinor fields, we need to combine spinors to obtain scalars.

12.2.1 ψ vs. $\psi^\dagger$

Spinors are complex, we need both ψ_L, ψ_R and $\psi_L^\dagger, \psi_R^\dagger$ to construct Hermitian Lagrangian. Notice that

$$\psi_{R,L}^\dagger \to \psi_{R,L}^\dagger e^{-\frac{1}{2}(i\theta_i \mp \beta_i)\sigma_i} \;\Rightarrow\quad \psi_R, \psi_L^\dagger \in V_{(0,\frac{1}{2})} \quad \psi_L, \psi_R^\dagger \in V_{(\frac{1}{2},0)}\,. \tag{12.5}$$

The overall sign has mathematical explanation. We have defined representation via a left group action $G \times V \to V$ with $h \cdot (g \cdot v) = (h \cdot g) \cdot v$. We can also use the right group action, which is equivalent: $V \times G \to V$, $(v \cdot g) \cdot h = v \cdot (g \cdot h)$. They are one to one correspondent. The left group action representation ρ satisfies $\rho(h) \circ \rho(g) = \rho(h \circ g)$, while the right group action presentation $\tilde{\rho}$ defined by $\tilde{\rho} := \rho(\cdot^{-1})$ satisfying $\tilde{\rho}\tilde{\rho}(g) = \tilde{\rho}(hg)$, i.e. $\psi_L^\dagger, \psi_R^\dagger$ transform in the $(0, \frac{1}{2}), (\frac{1}{2}, 0)$ irrep, implemented as a right action.

12.2.2 Tensor product representation

Products of spinors transform in tensor product representations

$$(\frac{1}{2}, 0) \times (\frac{1}{2}, 0) = (\frac{1}{2} \otimes \frac{1}{2}, 0 \times 0) = (0 \oplus 1, 0) = (0, 0) \oplus (1, 0)\,, \tag{12.6}$$

we can project $(\psi_L^\dagger, \psi_R)$, $(\psi_R^\dagger, \psi_L)$ onto trivial representation. If choose two different handed spinor, for example,

$$(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2} \otimes 0, 0 \otimes \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})\,, \tag{12.7}$$

this tell us that we can construct trivial representation by contracting with a four-vector from $(\psi_R^\dagger, \psi_R)$, $(\psi_L^\dagger, \psi_L)$.

12.2.3 The scalar representation from spinors

We consider transformation of $\psi_L^\dagger \psi_R = \sum_{i=1}^2 (\psi_L^\dagger)_i (\psi_R)_i$ where

$$\psi_R = \begin{pmatrix} (\psi_R)_1 \\ (\psi_R)_2 \end{pmatrix} = (\psi_R)_i e_i\,, \quad \psi_L = (\psi_L)_i e_i\,. \tag{12.8}$$

Remark 12.2. *$\{e_i \otimes e_j\}$ is a basis of $V_{(0,\frac{1}{2})} \otimes V_{(0,\frac{1}{2})}$, and the coefficient is related to $\psi_L^\dagger \psi_R$.*

The Lorentz transformation (in the variation form) is

$$\begin{aligned}\delta(\psi_L^\dagger \psi_R) &= \delta(\psi_L^\dagger)\psi_R + \psi_L^\dagger \delta(\psi_R) \\ &= -\frac{1}{2}(i\theta_i + \beta_i)\psi_L^\dagger \sigma_i \psi_R + \psi_L^\dagger \frac{1}{2}(i\theta_i + \beta_i)\sigma_i \psi_R = 0\,,\end{aligned} \tag{12.9}$$

which proves that $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$ is both Lorentz invariant and Hermitian,

12.2.4 The vector reps. from spinors

The four vectors of Lie algebra transforms as

$$\delta(v^0, v^i) = \left(\beta_i v^i, \beta_i v^0 - \varepsilon_{ijk}\theta_j v_k\right)\,. \tag{12.10}$$

We want to construct the corresponding basis of $V_{(\frac{1}{2},0)} \otimes V_{(0,\frac{1}{2})}$. Consider $\psi_R^\dagger \psi_R$

$$\begin{aligned}\delta\left(\psi_R^\dagger \psi_R\right) &= \delta(\psi_R^\dagger)\psi_R + \psi_R^\dagger \delta\psi_R \\ &= -\frac{1}{2}(i\theta_i - \beta_i)\psi_R^\dagger \sigma_i \psi_R + \frac{1}{2}(i\theta_i + \beta_i)\psi_R^\dagger \sigma_i \psi_R \\ &= \beta_i \psi_R^\dagger \sigma_i \psi_R\,.\end{aligned} \tag{12.11}$$

$\psi_R^\dagger \psi^R$ transform as the time component of a four vector. If we identify $\psi_R^\dagger \sigma_i \psi_R$ with the spatial component,

$$\begin{aligned}\delta(\psi_R^\dagger \sigma_i \psi_R) &= -\frac{1}{2}(i\theta_j - \beta_j)\psi_R^\dagger \sigma_j \sigma_i \psi_R + \frac{1}{2}(i\theta_j + \beta_j)\psi_R^\dagger \sigma_i \sigma_j \psi_R \\ &= \frac{1}{2} i\theta_j \psi_R^\dagger [\sigma_i, \sigma_j]\psi_R + \frac{1}{2}\beta_j \psi_R^\dagger \{\sigma_i, \sigma_j\}\psi_R \\ &= -\varepsilon_{ijk}\theta_j \psi_R^\dagger \sigma_k \psi_R + \beta_i \psi_R^\dagger \psi_R\,,\end{aligned} \tag{12.12}$$

which transforms as we expected. So $(\psi_R^\dagger \psi_R, \psi_R^\dagger \sigma_i \psi_R)$ transforms as a four vector. If we introduce $\sigma^\mu = (\mathbb{1}, \sigma_i)$, the four vector can be written as $\psi_R \sigma^\mu \psi_R$. Likewise $(\psi_L^\dagger \psi_L, -\psi_L^\dagger \sigma_i \psi_L)$ transform as (v^0, v^i), by introducing $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma_i)$, the four vector is $\psi_L^\dagger \bar{\sigma}^\mu \psi_L$.

12.3 Lorentz invariant Lagrangian involving spinors

12.3.1 The Dirac Lagrangian

We have the following proposals

- $\partial_\mu \psi_L^\dagger \partial^\mu \psi_R + \partial_\mu \psi_R^\dagger \partial^\mu \psi_L$, which is Lorentz invariant and Hermitian. Such a Lagrangian can be mapped to Lagrangian of multiple coupled scalar fields.
- $i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$, $i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L$, which is Lorentz invariant. And

$$(i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R)^\dagger = -i\partial_\mu \psi_R^\dagger (\sigma^\mu)^\dagger \psi_R = -i\partial_\mu \psi_R^\dagger \sigma^\mu \psi_R = -i\partial_\mu(\psi_R^\dagger \sigma^\mu \psi_R) + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R\,, \tag{12.13}$$

up to a total derivative it is also Hermitian. The mass term should be proportional to $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$. By dimension analysis, the action should be dimensionless $[S] = m^0$. By $\int d^4x \mathscr{L}$, $[\mathscr{L}] = m^4$. From kinetic term we know $[\psi] = m^{3/2}$, so $[\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L] = m^3$, the term should be

$$m\left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\right) . \tag{12.14}$$

Remark 12.3. *The mass term couples to ψ_L and ψ_R.*

For the theory with massive spinors, it is convenient to work with reducible Dirac representation, the Dirac spinor ψ defined as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} , \bar{\psi} = (\psi_R^\dagger, \psi_L^\dagger) , \tag{12.15}$$

where the $\bar{\psi}$ is called the Dirac conjugate. And we define the 4×4matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \tag{12.16}$$

using these notations, the Lagrangian can be written as a compact form,

$$\mathscr{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi . \tag{12.17}$$

Before we study the quantum theory, we first quickly learn the Clifford algebra for preparations.

12.3.2 The Clifford algebra

The γ^μ introduced above satisfy $\{\gamma^\mu, \gamma^\mu\} = 2g^{\mu\nu}\mathbb{1}$. Any realization of this algebra in therms of 4×4 matrices can be used to construct the Dirac representation. Define $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Purely involving the anticommutator, we can verify that

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho} S^{\mu\sigma} + g^{\mu\sigma} S^{\nu\rho} - g^{\nu\sigma} S^{\mu\rho} - g^{\mu\rho} S^{\nu\sigma}\right) , \tag{12.18}$$

which is the exactly the commutation relation between the generators of $so(1,3)$. $S^{\mu\nu}$ furnishes a representation of $so(1,3)$. If we choose

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} , \tag{12.19}$$

which is called the Weyl representation. The $S^{\mu\nu}$ can be written explicitly as

$$S^{ij} = \frac{1}{2}\varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} , \quad S^{0i} = -\frac{i}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} . \tag{12.20}$$

Compare to

$$\begin{aligned} \rho_{(0,\frac{1}{2})}(K_i) &= -\frac{i}{2}\sigma_i \\ \rho_{(\frac{1}{2},0)}(K_i) &= \frac{i}{2}\sigma_i . \end{aligned} \tag{12.21}$$

This reps decompose into $V_{(0,\frac{1}{2})} \oplus V_{(\frac{1}{2},0)}$. $(S^{\mu\nu})^\dagger$ to obtain $V_{(\frac{1}{2},0)} \oplus V_{(0,\frac{1}{2})}$. A second realization of the Clifford algebra,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \tag{12.22}$$

which is called the Majorana representation.

Remark 12.4. *At the level of representations, the Weyl and Majorana representation are equivalent.*

12.3.3 Lorentz transformation properties of spinor bilinearities revisited

We have $\Lambda_S = e^{i\theta_{\mu\nu}S^{\mu\nu}}$, which is the reducible $(0,\frac{1}{2}) \otimes (\frac{1}{2},0)$ representation. In another side, we have $\Lambda_\nu = e^{i\theta_{\mu\nu}V^{\mu\nu}}$, the irreducible 4-vector representation. How to construct Lorentz scalar from product of Dirac spinors? First, we claim that $(\gamma^0\Lambda_S\gamma^0)^\dagger = \Lambda_S^{-1}$ and $(\gamma^0)^2 = \mathbb{1}$. Give the claim, for a Dirac spinor ψ,

$$\psi^\dagger\gamma^0 \to (\Lambda_S\psi)^\dagger\gamma^0 = \psi^\dagger\gamma^0\gamma^0\Lambda_S^\dagger\gamma^0 = \psi^\dagger\gamma^0\left(\gamma^0\Lambda_S\gamma^0\right)^\dagger = \psi^\dagger\gamma^0\Lambda_S^{-1}, \tag{12.23}$$

where we use $(\gamma^0)^\dagger = \gamma^0$, so $\psi^\dagger\gamma^0\psi$ transform as a scalar.

Lemma 12.1.

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 . \tag{12.24}$$

Proof: From the anticommutator $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$, we know that

$$(\gamma^0)^2 = \mathbb{1}, \quad (\gamma^i)^0 = -\mathbb{1}, \tag{12.25}$$

which means that the eigenvalue of γ^0 is ± 1 and the eigenvalue of γ^i is $\pm i$. If the Clifford algebra is realized in terms of normal and unitary diagonalizable matrices, hence

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \tag{12.26}$$

i.e.

$$\begin{aligned} (\gamma^0)^\dagger &= \gamma^0\gamma^0\gamma^0, \\ (\gamma^i)^\dagger &= \gamma^0\gamma^i\gamma^0. \end{aligned} \tag{12.27}$$

Lemma 12.2.

$$(S^{\mu\nu})^\dagger = \gamma^0 S^{\mu\nu}\gamma^0, \tag{12.28}$$

Proof:

$$(S^{\mu\nu})^\dagger = \left(\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)^\dagger = -\frac{i}{4}[\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0] = -\frac{i}{4}\gamma^0[\gamma^\nu, \gamma^\nu]\gamma^0 = \gamma^0 S^{\mu\nu}\gamma^0 . \tag{12.29}$$

We then prove the claim $(\gamma^0\Lambda_S\gamma^0)^\dagger = \Lambda_S^{-1}$ now,
Proof:

$$(\gamma^0\Lambda_S\gamma^0)^\dagger = \left(\gamma^0 e^{i\theta_{\mu\nu}S^{\mu\nu}}\gamma^0\right)^\dagger = \gamma^0 e^{-i\theta_{\mu\nu}(S^{\mu\nu})^\dagger}\gamma^0 = e^{-i\theta_{\mu\nu}S^{\mu\nu}} = \Lambda_S^{-1} . \tag{12.30}$$

The next question is how to construct Lorentz 4-vectors from Dirac spinor bilinears. Using the property

$$\Lambda_S^{-1}\gamma^\mu\Lambda_S = (\Lambda_V)^\mu{}_\nu\gamma^\nu, \tag{12.31}$$

It is easy to guess $\psi^\dagger\gamma^0\gamma^\mu\psi$ transforms as

$$\psi^\dagger\gamma^0\gamma^\mu\psi \to \psi^\dagger\gamma^0\Lambda_S^{-1}\gamma^\mu\Lambda_S\psi = (\Lambda_V)^\mu{}_\nu\psi^\dagger\gamma^0\gamma^\nu\psi\,, \tag{12.32}$$

which is indeed a Lorentz 4-vector.

Remark 12.5. *with $\psi = (\psi_L, \psi_R)^T$ this reproduces the results obtained above via the Weyl representation.*

12.4 The Dirac equation

12.4.1 The free equation

The Lagrangian is

$$\mathscr{L} = \psi^\dagger\gamma^0(i\gamma^\mu\partial_\mu - m)\psi = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,, \tag{12.33}$$

where $A\!\!\!/ := \gamma^\mu A_\mu$. The EoM ($\psi$ and $\psi^\dagger$ are independent)

$$\begin{gathered}\text{for}\quad \psi: \partial_\mu\frac{\partial\mathscr{L}}{\partial_\mu\psi} - \frac{\partial\mathscr{L}}{\partial\psi} = 0 \quad\Rightarrow\quad \partial_\mu(\bar{\psi}i\gamma^\mu) + m\bar{\psi} = 0\,,\\ \Leftrightarrow\quad i\partial_\mu(\psi^\dagger\gamma^0\gamma^\mu)\gamma^0 + m\psi^\dagger\gamma^0\gamma^0 = 0 \quad\Leftrightarrow\quad (-i\partial\!\!\!/\psi + m\psi)^\dagger = 0\,.\end{gathered} \tag{12.34}$$

For $\psi^\dagger$, we rewrite the Lagrangian as $\mathscr{L} = -i\partial_\mu\bar{\psi}\gamma^\mu\psi + i\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) - m\bar{\psi}\psi$, then

$$\partial_\mu\frac{\partial\mathscr{L}}{\partial_\mu\psi^\dagger} - \frac{\partial\mathscr{L}}{\partial\psi^\dagger} = 0 \quad\Leftrightarrow\quad -i\partial\!\!\!/\psi + m\psi = 0\,. \tag{12.35}$$

We then get the Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0\,. \tag{12.36}$$

Remark 12.6. *Each component of ψ satisfies ordinary Klein-Gordan equation, to see that, acting $(i\partial\!\!\!/+m)$ on both sides of the Dirac equation,*

$$\begin{gathered}(i\partial\!\!\!/ + m)(i\partial\!\!\!/ - m)\psi = 0 \quad\Leftrightarrow\quad (-\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu - m^2)\psi = 0\\ \Leftrightarrow\quad (\Box + m^2)\psi = 0\,,\end{gathered} \tag{12.37}$$

which is the dispersion relation $p^2 = m^2$ upon Fourier transformation.

12.4.2 Coupling Dirac spinors to the photon: the QED Lagrangian

We have ψ transform in representation (parameterized by $\mathbb{Z}$) of $U(1)$ (gauge) group,

$$\psi \mapsto e^{-i\omega}\psi\,, \tag{12.38}$$

which is a global symmetry of free Dirac Lagrangian. Evaluate to a local symmetry,

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu\,, \tag{12.39}$$

such that

$$D_\mu\psi = e^{-i\omega}D_\mu\psi\,. \tag{12.40}$$

We then have QED Lagrangian,

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi\,. \tag{12.41}$$

The Equation of motion is, for ψ,

$$(i\not{D} - m)\psi = 0 \quad \Leftrightarrow \quad (i\not{\partial} - e\not{A} - m)\psi = 0\,, \tag{12.42}$$

which is the Dirac equation of spinor minimally coupled to a photon. EoM for A^μ,

$$\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi\,, \tag{12.43}$$

which is the Maxwell equation with source $J^\mu = e\bar{\psi}\gamma^\mu\psi$.

12.5 Quantum Dirac spinor fields

12.5.1 Review of setup

$\psi(x)$ is a quantum Dirac spinor field, which transforms under the Lorentz transformation as

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \mathscr{D}_{\text{Dirac}}(\Lambda^{-1})\psi(\Lambda x) = \Lambda_S^{-1}\psi(\Lambda x)\,, \tag{12.44}$$

where

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_\sigma \left(u_\sigma(p)a_\sigma(p)e^{-ipx} + v_\sigma(p)b_\sigma^\dagger e^{ipx}\right)\,, \tag{12.45}$$

where $u_\sigma(p), v_\sigma(p)$ are polarization tensor, a_p is the annihilation operator for particle and $b_p^\dagger$ is the creation operator for anti-particle. In order to make it well defined, we need to clarify two things

- identify infinite dimension unitary representation of Lorentz group.
- find the $u_\sigma(p), v_\sigma(p)$.

12.5.2 What type of particles can Dirac field package

Recall that the little group for massive particle is $SO(3)$, which is the subgroup of $SO^\uparrow(1,3)$, where we keep the fiducial momentum $k = (m,0,0,0)$ fixed. Now

$$\begin{array}{ccc} SL(2,\mathbb{C}) & \xrightarrow{2:1} & SO^\uparrow(1,3) \\ \supset\big\downarrow & & \big\downarrow\supset \\ SU(2) & \xrightarrow{2:1} & SO(3) \end{array}\,, \tag{12.46}$$

and $k_\mu\sigma^\mu = m\sigma^0$ is fixed. Decompose $(\frac{1}{2},0)\oplus(0,\frac{1}{2})|_{SU(2)}$ (irreps of $SU(2)$) and recall that $SU(2)$ (of rotations) can embedded diagonally into $SU(2)\oplus SU(2)$,

$$(\frac{1}{2},0)\oplus(0,\frac{1}{2})|_{SU(2)} = \frac{1}{2}\otimes 0\oplus 0\otimes\frac{1}{2} = \frac{1}{2}\oplus\frac{1}{2}\,. \tag{12.47}$$

The only possible choice of irreps for the little group is the spin $\frac{1}{2}$ irrep.

12.5.3 The intertwiner solve the Dirac equation

Based on the discussion above, $u(p)$ ($v(p)$) takes the following form

$$u(p) = \begin{pmatrix} u^1_{1/2} & u^1_{-1/2} \\ u^2_{1/2} & u^2_{-1/2} \\ u^3_{1/2} & u^3_{-1/2} \\ u^4_{1/2} & u^4_{-1/2} \end{pmatrix}, \tag{12.48}$$

where the row index is the spin index and the column index is the Dirac index of $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ reps of $SL(2,\mathbb{C})$. Likewise, $v(p)$ has the similar form. u and v must satisfy (recall the equation (8.32))

$$\begin{aligned} \mathscr{D}(\Lambda)u(p) &= \Lambda_S u(p) = u(\Lambda p)D_{\frac{1}{2}}(W(\Lambda,p))\,, \\ \mathscr{D}(\Lambda)v(p) &= \Lambda_S v(p) = v(\Lambda p)D^*_{\frac{1}{2}}(W(\Lambda,p))\,. \end{aligned} \tag{12.49}$$

Proof: These two equations need some more details. The initial equation for polarization tensor is

$$\sum_\sigma D^*_{\sigma'\sigma}(W(\Lambda,p))\varepsilon^\mu{}_\sigma(p) = \sum_\nu \mathscr{D}(\Lambda^{-1})^\mu{}_\nu\varepsilon^\nu{}_{\sigma'}(\Lambda p)\,. \tag{12.50}$$

By definition of Hermitian conjugation, we have

$$\sum_\sigma D^\dagger_{\sigma\sigma'}(W(\Lambda,p))\varepsilon^\mu{}_\sigma(p) = \sum_\nu \mathscr{D}(\Lambda^{-1})^\mu{}_\nu\varepsilon^\nu{}_{\sigma'}(\Lambda p)\,. \tag{12.51}$$

Changing $\Lambda\to\Lambda^{-1}$, $p\to\Lambda p$, we have

$$\sum_\sigma D^\dagger_{\sigma\sigma'}(W(\Lambda^{-1},\Lambda p))\varepsilon^\mu{}_\sigma(\Lambda p) = \sum_\nu \mathscr{D}(\Lambda)^\mu{}_\nu\varepsilon^\nu{}_{\sigma'}(p)\,. \tag{12.52}$$

Taking the advantage of the unitarity of D, and write the equation in matrix form, we have

$$\mathscr{D}(\Lambda)\varepsilon(p) = \varepsilon(\Lambda p)D_J(W(\Lambda^{-1},\Lambda p)^{-1})\,, \tag{12.53}$$

where J denotes the spin-J representation. Based on the definition of the operator W,

$$W(\Lambda^{-1},\Lambda p)^{-1} = U(L^{-1}(p)\Lambda^{-1}L(\Lambda p))^{-1} = U(L^{-1}(\Lambda p)\Lambda L(p)) = W(\Lambda,p)\,, \tag{12.54}$$

substitute back to the matrix equation, we have

$$\mathscr{D}(\Lambda)\varepsilon(p) = \varepsilon(\Lambda p) D_J(W(\Lambda, p))\,, \tag{12.55}$$

back to our discussion, we have

$$\Lambda_S u(p) = u(\Lambda p) D_{\frac{1}{2}}(W(\Lambda, p))\,, \quad \Lambda_S v(p) = v(\Lambda p) D^*_{\frac{1}{2}}(W(\Lambda, p))\,. \tag{12.56}$$

It is convenient to denote

$$u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}\,, \quad v = \begin{pmatrix} v^+ \\ v^- \end{pmatrix}\,, \tag{12.57}$$

where $u^\pm, v^\pm$ are all 2×2 matrices. Consider first $p = k$, then Λ is a spatial rotation ρ, such that $W(\rho, k) = \rho$. Recall that with

$$\omega_{ij} = \begin{pmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{pmatrix}\,, \quad D_{\frac{1}{2}}(\rho) = e^{\frac{i}{2}\varepsilon_{ijk}\omega_{ij}\sigma_k} = e^{i(\theta_x\sigma_1 + \theta_y\sigma_2 + \theta_z\sigma_3)}\,. \tag{12.58}$$

$$\rho_S = e^{i\omega_{ij}S^{ij}}\,, \quad S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}\,, \tag{12.59}$$

the equation reduces to

$$\sigma_k u^\pm(k) = u^\pm(k)\sigma_k\,, \tag{12.60}$$

which means that $u^\pm(k)$ commutes with all the generator of $SU(2)$. By Schur's lemma,

$$u^\pm(k) = c^\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{12.61}$$

For $v(\sigma)$, the equation is

$$\sigma_i v^\pm(k) = -v^\pm(k)\sigma_i^* \quad \Leftrightarrow \quad \sigma_i v^\pm(k)\sigma_2 = v^\pm(k)\sigma_2\sigma_i\,, \tag{12.62}$$

where we use $\sigma_i^* = -\sigma_2\sigma_i\sigma_2$ and $\sigma_2^2 = \mathbb{1}$. Then $[v^\pm(k)\sigma_2, \sigma_i] = 0$, by Schur's lemma,

$$v^\pm(k)\sigma_2 = d^\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow v^\pm(k) = d^\pm \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,. \tag{12.63}$$

$u^\pm(k)$ and $v^\pm(k)$ solve intertwines equation for any choice of $c^\pm, d^\pm$. Parity and locality (as we will see) impose $c^+ = c^-$, $d^+ = -d^-$. A choice of normalization is $c^\pm = \sqrt{m}$, $d^\pm = \mp i\sqrt{m}$,

$$u(k) = \sqrt{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \quad v(k) = \sqrt{m} \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}\,. \tag{12.64}$$

For arbitrary momentum p, we know that

$$u(p) = L(p)_S u(k)\,, \quad v(p) = L(p)_S v(k)\,, \tag{12.65}$$

where $L(p)$ is the fiducial Lorentz transformation $L(p)k = p$. With

$$L(p)^{\mu}{}_{\nu}\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}^{\nu} = \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}^{\mu}\,, \quad L(p) = \begin{pmatrix} \frac{p^0}{m} & \frac{p^1}{m} & \frac{p^2}{m} & \frac{p^3}{m} \\ \frac{p^1}{m} & & & \\ \frac{p^2}{m} & & & \\ \frac{p^3}{m} & & & \end{pmatrix}. \tag{12.66}$$

Recall that $\Lambda_S^{-1}\gamma^{\mu}\Lambda_S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$, $\Lambda_S^{-1}\gamma^0 = \Lambda^0{}_{\nu}\gamma^{\nu}\Lambda_S^{-1}$. In Weyl realization of γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \Rightarrow \gamma^0 u(k) = u(k)\,, \gamma^0 v(k) = -v(k)\,, \tag{12.67}$$

$$\Rightarrow \quad u(p) = L(p)_S u(k) = L(p)_S \gamma^0 u(k) = (L(p)^{-1})^0{}_{\nu}\gamma^{\nu}L(p)_S u(k)\,, \tag{12.68}$$

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda^{\rho}{}_{\sigma}g_{\rho\nu}g^{\sigma\mu} \quad \Rightarrow \quad (L(p)^{-1})^0{}_{\nu} = (L(p))^{\rho}{}_{\sigma}g_{\rho\nu}g^{\sigma 0} = (L(p))^{\rho 0}g_{\rho\nu} = \frac{p^{\rho}}{m}g_{\rho\nu} = \frac{p_{\nu}}{m}\,. \tag{12.69}$$

This gives

$$u(p) = \frac{p_{\nu}\gamma^{\nu}}{m}u(p) \quad \Rightarrow \quad (\not{p} - m)u(p) = 0\,. \tag{12.70}$$

Likewise

$$v(p) = -\frac{\not{p}}{m}v(p) \quad \Rightarrow \quad (\not{p} + m)v(p) = 0\,. \tag{12.71}$$

In this case, the quantum Dirac spinor field satisfies $(i\not{\partial} - m)\psi(x) = 0$.

Remark 12.7. *Pay attention the difference between the previous discussion. In the former section we derive the EoM for $\psi(x)$, which is the same. However, it is from Lagrangian and isn't quantized. Now the quantized field still satisfies the EoM of the same form, but have different meaning.*

12.6 Chirality and the γ^5 matrix

Chirality distinguishes between two Weyl irreps, $(\frac{1}{2}, 0)$ left-handed and $(0, \frac{1}{2})$ right-handed. Recall that a Dirac spinor has no fixed chirality, we can project it onto the chiral subspace via γ^5 matrix.

Definition 12.2.

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{12.72}$$

Theorem 12.2. $\left\{\frac{1\pm\gamma^5}{2}\right\}$ *is a complete set of projection operator.*

Proof: The proof separates into two steps. First, we prove that $\frac{1\pm\gamma^5}{2}$ is a projector, since

$$\left(\frac{1\pm\gamma^5}{2}\right)^2 = \frac{1 \pm 2\gamma^5 + (\gamma^5)^2}{4} = \frac{1\pm\gamma^5}{2}\,, \tag{12.73}$$

where we used $(\gamma^5)^2 = \mathbb{1}$. And

$$\frac{1 \pm \gamma^5}{2} \frac{1 \mp \gamma^5}{2} = \frac{1}{4} \left(1 + \gamma^5 - \gamma^5 - (\gamma^5)^2\right) = 0\,. \tag{12.74}$$

These projector will project the states onto γ^5 eigenspace to eigenvalue ± 1. Because $\operatorname{Tr} \gamma^5 = i \operatorname{Tr}\left(\gamma^0 \gamma^1 \gamma^2 \gamma^3\right) = 0$, the number of the two eigenvalue ± 1 must be the same. So the dimension of each eigenspace is 2.

Theorem 12.3. *Eigenspace of γ^5 is invariant under $SL(2,\mathbb{C})$ transformation.*

Proof: Consider the generator of $SL(2,\mathbb{C})$ transformation $S^{\mu\nu}$ and

$$S^{\mu\nu} \frac{1 \pm \gamma^5}{2} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] \frac{1 \pm \gamma^5}{2} \overset{\{\gamma^5, \gamma^\mu = 0\}}{=} \frac{1 \pm \gamma^5}{2} \frac{i}{4} [\gamma^\mu, \gamma^\nu] = \frac{1 \pm \gamma^5}{2} S^{\mu\nu}\,. \tag{12.75}$$

We conclude that the eigenspaces are irreducible $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ subspaces of the rep. $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

Chapter 13

Discrete symmetries and CPT theorem

13.1 Beyond $SO^{\uparrow}(1,3)$

Recall that $SO^{\uparrow}(1,3)$ is the connected component of $O(1,3)$, which contains the identity,

$$SO^{\uparrow}(1,3) = \operatorname{Ker}\varphi_{\det} \cap \operatorname{Ker}\varphi_{sgn}\,. \tag{13.1}$$

What about the other 3 components, obtained by adjoining

$$\text{space inversion}\quad \mathbb{P} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \text{time reversal}\quad \mathbb{T} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{13.2}$$

and $\mathbb{P}\mathbb{T}$ to $SO^{\uparrow}(1,3)$. These may or may not be symmetries of fundamental interactions. If they are, those should exist operation $P = U(\mathbb{P})$ and $T = U(\mathbb{T})$ acting on the Hilbert space which acts as

$$\begin{aligned} PU(a,\Lambda)P^{-1} &= U(\mathbb{P}a, \mathbb{P}\Lambda\mathbb{P}^{-1})\,, \\ TU(a,\Lambda)T^{-1} &= U(\mathbb{T}a, \mathbb{T}\Lambda\mathbb{T}^{-1})\,. \end{aligned} \tag{13.3}$$

13.2 Action of P, T on Poincaré generator

13.2.1 Parity

Recall $U(a,1) = e^{ia_\mu \hat{P}^\mu}$, where $\hat{P}^\mu = (\hat{H}, \hat{P}^i)$ (we add the hat to distinguish with the representation of the space inversion P.).

$$\begin{aligned} PU(a,1)P^{-1} = P\left(\mathbb{1} + ia_\mu \hat{P}^\mu + \dots\right)P^{-1} &= \mathbb{1} + ia_\mu P\hat{P}^\mu P^{-1} + \dots \\ &\stackrel{!}{=} U(\mathbb{P},1) = 1 + i(\mathbb{P}a)\cdot\hat{P} + \dots\,, \end{aligned} \tag{13.4}$$

which gives

$$Pi\hat{P}P^{-1} = i\mathbb{P}\hat{P}\,. \tag{13.5}$$

If P is linear, unitary (Wigner's theorem), we have

$$P\hat{P}P^{-1} = \mathbb{P} \cdot \hat{P} \quad \Leftrightarrow \quad \begin{cases} P\hat{H}P^{-1} = H \\ P\hat{P}^i P^{-1} = -\hat{P}^i \end{cases} . \tag{13.6}$$

To study the action on Lorentz generator, note that for $\forall \Lambda \in SO^{\uparrow}(1,3)$,

$$\begin{aligned} \mathbb{P}\Lambda\mathbb{P}^{-1} &\in SO^{\uparrow}(1,3)\,, \\ \mathbb{T}\Lambda\mathbb{T}^{-1} &\in SO^{\uparrow}(1,3)\,. \end{aligned} \tag{13.7}$$

$\mathbb{P}$ and $\mathbb{T}$ act via conjugation on Lie algebra $so(1,3)$. For $U(a,\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}\right) = U(\Lambda)$, where

$$\Lambda^{\rho}{}_{\sigma} = \delta^{\rho}_{\sigma} + \omega^{\rho}{}_{\sigma} + \dots\,, \tag{13.8}$$

and

$$V^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix} , \tag{13.9}$$

we have

$$\begin{aligned} PU(\Lambda)P^{-1} &= P\left(1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + \dots\right)P^{-1} \\ &\overset{!}{=} U(\mathbb{P}\Lambda\mathbb{P}^{-1}) = 1 + i(\mathbb{P}\omega\mathbb{P}^{-1})_{\mu\nu}\hat{J}^{\mu\nu} + \dots \\ \Rightarrow \quad P\hat{J}^{\mu\nu}P^{-1} &= (\mathbb{J}\hat{J}\mathbb{P})^{\mu\nu} \quad \Leftrightarrow \quad \begin{cases} P\hat{K}^i P^{-1} = -\hat{K}^i \\ P\hat{J}^i P^{-1} = \hat{J}^i \end{cases} , \end{aligned} \tag{13.10}$$

for $i = 1, 2, 3$.

13.3 Time reversal

Similar to the parity operator, denoting $T = U(\mathbb{T})$, we should get

$$TU(a,1)T^{-1} = U(\mathbb{T}a, 1) \quad \Rightarrow \quad Ti\hat{P}T^{-1} = i\mathbb{T} \cdot \hat{P}\,. \tag{13.11}$$

Assume T is linear, $T\hat{P}T^{-1} = \mathbb{T} \cdot \hat{P} \Rightarrow T\hat{H}T^{-1} = -H$. If T is a symmetry, for any state $|\psi\rangle$ of energy E, $\exists\, |\phi\rangle = T\,|\psi\rangle$ such that $H\,|\phi\rangle = HT\,|\psi\rangle = -TH\,|\psi\rangle = -E\,|\phi\rangle$, leading to the negative energy solution. So the only possible choice is that T is anti-linear. Assume that T is anti-linear

$$Ti\hat{P}T^{-1} = i\mathbb{T} \cdot \hat{P} \quad \Rightarrow \quad -iT\hat{P}T^{-1} = i\mathbb{T} \cdot \hat{P} \quad \Rightarrow \quad \begin{cases} T\hat{H}T^{-1} = H \\ T\hat{P}^i T^{-1} = -\hat{P}^i \\ T\hat{K}^i T^{-1} = \hat{K}^i \\ T\hat{J}^i T^{-1} = -\hat{J}^i \end{cases} . \tag{13.12}$$

13.4 Action of P, T on Hilbert space

Exemplified: on parity and massive representation of $SO^{\uparrow}(1,3)$ or $SL(2,\mathbb{C})$. Consider $|\psi_{k,\sigma}\rangle$, where $k = (m,0,0,0)$ in a given irrep of $SO(3)$ or $SU(2)$,

$$\begin{aligned}\hat{P}^\mu |\psi_{k,\sigma}\rangle &= k^\mu |\psi_{k,\sigma}\rangle \ , \\ \hat{J}_3 |\psi_{k,\sigma}\rangle &= \sigma |\psi_{k,\sigma}\rangle \ .\end{aligned} \tag{13.13}$$

Then

$$\hat{P}^\mu P |\psi_{k,\sigma}\rangle = P(\mathbb{P}\cdot\hat{P})^\mu |\psi_{k,\sigma}\rangle = P(\mathbb{P}\cdot k)^\mu |\psi_{k,\sigma}\rangle \overset{\vec{k}=0}{=} k^\mu P |\psi_{k,\sigma}\rangle \ , \tag{13.14}$$

which gives the same momentum.

$$\hat{J}_3 P |\psi_{k,\sigma}\rangle = P\hat{J}_3 |\psi_{k,\sigma}\rangle = \sigma P |\psi_{k,\sigma}\rangle \ . \tag{13.15}$$

If we assume that the joint eigenspaces are non-degenerate, $P|\psi_{k,\sigma}\rangle = \eta |\psi_{k,\sigma}\rangle$ with $|\eta| = 1$ (up to a phase). Is there any σ-dependence in η? Notice that the raising and lowering operator $\hat{J}_1 \pm i\hat{J}_2$ of σ commute with P, η is σ independent. Is there any momentum dependence of η? Consider

$$|\psi_{p,\sigma}\rangle = U(L(p))|\psi_{k,\sigma}\rangle \quad \Rightarrow \quad P|\psi_{p,\sigma}\rangle = PU(L(p))P^{-1}P|\psi_{k,\sigma}\rangle = U\left(\mathbb{P}L(p)\mathbb{P}^{-1}\right)\eta_k |\psi_{k,\sigma}\rangle \ . \tag{13.16}$$

Lemma 13.1.

$$\mathbb{P}L(p)\mathbb{P}^{-1} = L(\mathbb{P}p) \, . \tag{13.17}$$

Proof: The $L(p)$ can be decomposed into the boost along the z-axis and the rotation that rotates z-axis to the direction of arbitrary spatial momentum $\vec{p}$,

$$L(p) = R(e_z \to \vec{p}) \circ B(\vec{0} \to |\vec{p}|e_z) \, , \tag{13.18}$$

then

$$\begin{aligned}\mathbb{P}L(p)\mathbb{P}^{-1} &= \mathbb{P}R(e_z \to \vec{p})\mathbb{P}\mathbb{P}^{-1}B(\vec{0} \to |\vec{p}|e_z)\mathbb{P}^{-1} = R(e_z \to \vec{p})B(\vec{0} \to -|\vec{p}|e_z) \\ &= R(e_z \to -\vec{p})B(\vec{0} - |\vec{p}|e_z) = L(\mathbb{P}p) \, ,\end{aligned} \tag{13.19}$$

where we have already used the result that the parity operator commutes with the generators of rotations and anti-commutes with the generators of boosts.

Then the equation (13.16) becomes

$$P|\psi_{p,\sigma}\rangle = U\left(\mathbb{P}L(p)\mathbb{P}^{-1}\right)\eta_k |\psi_{k,\sigma}\rangle = U(L(\mathbb{P}p))\eta_k |\psi_{k,\sigma}\rangle = \eta_k |\psi_{\mathbb{P}p,\sigma}\rangle \ , \tag{13.20}$$

which means that η is momentum independent. η is called the intertwine parity of the particle $|\psi_{k,\sigma}\rangle$. Note that $P^2|\psi_{p,\sigma}\rangle = P\eta_k|\psi_{\mathbb{P}p,\sigma}\rangle = \eta^2|\psi_{p,\sigma}\rangle$, P^2 is called the internal symmetry (not related to the spacetime), which commutes with all Poincaré generators.

Example 13.1. *The standard model has the conserved charges baryon number $Q_B(Q_1)$, lepton number $Q_L(Q_2)$ and electromagnetic charge $Q_{em}(Q_3)$, $e^{i\sum\alpha_i Q_i}$ for $\alpha_i \in [0, 2\pi)$ is an internal symmetry. P is not uniquely defined as we see in the example. For given P, we can define $P' = Pe^{i\sum\alpha_i Q_i}$ satisfies the same relation of parity, but yield different numerical value for the intrinsic parity of particles. The intrinsic* *parity of* 3 *independent particles can be fixed by choosing* α_i *appropriately.*

The time reversal T is similar, except the occurring phase can be absorbed in the definition of the state (due to the anti-linear of T).

13.5 Transformation of quantum fields under the discrete symmetries $\mathbb{P}$ and $\mathbb{T}$

We exemplified on $\mathbb{P}$ and massive irreps.

13.5.1 Transformation of creation and annihilation operator

The creation operator

$$a^\dagger_{p,\sigma}|0\rangle = \frac{1}{\sqrt{2\omega_p}}|\psi_{p,\sigma}\rangle \ , \tag{13.21}$$

which gives

$$\frac{1}{\sqrt{2\omega_p}}P|\psi_{p,\sigma}\rangle = \frac{1}{\sqrt{2\omega_p}}\eta|\psi_{\mathbb{P}p,\sigma}\rangle = \eta a^\dagger_{\mathbb{P}p,\sigma}|0\rangle = Pa^\dagger_{p,\sigma}P^{-1}P|0\rangle \ , \tag{13.22}$$

if we assume the vacuum is parity invariant, $P|0\rangle = |0\rangle$, we conclude that

$$\begin{aligned} Pa^\dagger_{p,\sigma}P^{-1} &= \eta a^\dagger_{\mathbb{P}p,\sigma} \\ \Rightarrow \quad Pa_{p,\sigma}P^{-1} &= \eta^* a_{\mathbb{P}p,\sigma} \, . \end{aligned} \tag{13.23}$$

13.5.2 Complex scalar field

The parity transformation acting on the field,

$$\begin{aligned} P\phi(x)P^{-1} &= P\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[a_p e^{-ipx} + b^\dagger_p e^{ipx}\right]P^{-1} \\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(\eta^* a_{\mathbb{P}p}e^{-ipx} + \eta^c b^\dagger_{\mathbb{P}p}e^{ipx}\right) \ , \end{aligned} \tag{13.24}$$

where η^c is the intrinsic parity of anti-particle. By imposing $\eta^c = \eta^*$ to protect the theory, we have

$$\eta^*\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-i(\mathbb{P}p)x} + b^\dagger_p e^{i(\mathbb{P}p)x}\right) = \eta^*\phi(\mathbb{P}x)\,, \tag{13.25}$$

where we use the invariant measure property, $\omega_p = \omega_{\mathbb{P}p}$,, $\mathbb{P}^{-1} = \mathbb{P}$ and change the integration variable.

Remark 13.1. *In a theory conserving parity, bound state of a scalar particle and its antiparticle mush have parity* $\eta\eta^c = \eta\eta^* = 1$.

13.5.3 Dirac spinor field

Using the same relation (13.23), we have

$$P\psi(x)P^{-1} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[\eta^* u_\sigma(\mathbb{P}p)a_\sigma(p)e^{ip\mathbb{P}x} + \eta^c v_\sigma(\mathbb{P}p)b^\dagger_\sigma(p)e^{ip\mathbb{P}x}\right] \, . \tag{13.26}$$

By $\gamma^0 S^{0i}\gamma^0 = -S^{0i}$, $\gamma^0 S^{ij}\gamma^0 = S^{ij}$,

$$\Rightarrow \quad \mathscr{D}_{\text{Dirac}}(\mathbb{P}\Lambda\mathbb{P}^{-1}) = \gamma^0 \mathscr{D}_{\text{Dirac}}(\Lambda)\gamma^0\,, \tag{13.27}$$

$$\begin{aligned} \Rightarrow \quad u_\sigma(\mathbb{P}p) &= \mathscr{D}_{\text{Dirac}}\left(L(\mathbb{P}p)\right) u_\sigma(k) = \mathscr{D}_{\text{Dirac}}(\mathbb{P}L(p)\mathbb{P}^{-1})u_\sigma(k) \\ &= \mathscr{D}_{\text{Dirac}}(\mathbb{P}L(p)\mathbb{P}^{-1})u_\sigma(k) = \gamma^0 \mathscr{D}_{\text{Dirac}}(L(p))\gamma^0 u_\sigma(k)\,, \end{aligned} \tag{13.28}$$

where used the lemma (13.1). Notice that $\gamma^0 u_\sigma(k) = u_\sigma(k)$, the equation reduces to $\gamma^0 u_\sigma(p)$. Likewise $v_\sigma(\mathbb{P}p) = -\gamma^0 v_\sigma(k)$, where the minus sign comes from $\gamma^0 v_\sigma(k) = -v_\sigma(k)$. We conclude that

$$P\psi(x)P^{-1} = \eta^* \gamma^0 \psi(\mathbb{P}x)\,, \tag{13.29}$$

where we impose $\eta^c = -\eta^*$. In a theory conserving parity, the intrinsic parity of one survive bound state of a Dirac spinor and antiparticle is $\eta\eta^c = -|\eta|^2 = -1$. Recall that

$$u(k) = \begin{pmatrix} c^+ & 0 \\ 0 & c^- \\ c^- & 0 \\ 0 & c^- \end{pmatrix}\,, \quad v(k) = i \begin{pmatrix} 0 & -d^+ \\ d^+ & 0 \\ 0 & -d^- \\ d^- & 0 \end{pmatrix}\,. \tag{13.30}$$

We need $\gamma^0 u(k) = \alpha u(k)$,

$$\begin{aligned} \gamma^0 u(k) = \begin{pmatrix} c^- & 0 \\ 0 & c^- \\ c^+ & 0 \\ 0 & c^+ \end{pmatrix} \quad &\Rightarrow \quad \alpha c^+ = c^-\,, \alpha c^- = c^+ \\ &\Leftrightarrow \quad \alpha^2 = 1, \alpha = \pm 1, c^+ = \pm c^-\,. \end{aligned} \tag{13.31}$$

Likewise $\gamma^0 v(k) = \beta v(k)$,

$$\begin{aligned} \gamma^0 v(k) = \begin{pmatrix} 0 & -d^- \\ d^- & 0 \\ 0 & -d^+ \\ d^+ & 0 \end{pmatrix} &\Rightarrow d^+ = \beta d^-, d^- = \beta d^+\,, \\ &\Rightarrow \quad \beta^= 1, \beta = \pm 1, d^+ = \pm d^-\,. \end{aligned} \tag{13.32}$$

13.6 Charge conjugation

Processes which are not invariant under $\mathbb{P}$, $\mathbb{T}$ transformation, are invariant under $\mathbb{PT}$ if we also exchange particle and anti particles. We introduce the operator $\hat{C}$, such that

$$\begin{aligned} \hat{C}U(a,\Lambda)\hat{C}^{-1} &= U(a,\Lambda) \\ \hat{C}a^\dagger_\sigma(p)\hat{C}^{-1} &= \xi b^\dagger_\sigma(p)\,, \end{aligned} \tag{13.33}$$

where ξ is the intrinsic phase. Similar analysis as above yields,

$$\begin{aligned} \hat{C}\phi(x)\hat{C}^{-1} &= \xi^* \phi^*(x)\,, \\ \hat{C}\psi(x)\hat{C}^{-1} &= -\xi^* i\gamma^2 \psi^*(x)\,. \end{aligned} \tag{13.34}$$

13.7 CPT invariance

All Lorentz invariant interactions that we can construct from quantum fields lead to CPT invariant actions.

Chapter 14

Spin statistics

14.1 What types of statistic are possible?

Before we start the discussion, we need to clarify what is statistic.

Definition 14.1. *Statistics is identical particle behaviour of a product state under exchange of identical particles.*

Why the name partition function in statistical physics depends on statistics of particles? Recall that a particle is specified by (geometrically) irreducible representation of the Poincaré group and a full set of quantum numbers (for example, the electric charge) excluding the Poincaré quantum numbers. $|\vec{p}, s, n\rangle$, where $\vec{p}$ is the momentum, s is the spin/helicity index, n is the special label. The state describing two identical free particles of species will reside in the Hilbert space $\mathscr{H}_1^n \otimes \mathscr{H}_1^n$, where n is the special label and subscript 1 means the one particle Hilbert space.

$$\mathscr{H}_1^n \otimes \mathscr{H}_1^n = \langle |\vec{p}_1, s_1; n\rangle \otimes |\vec{p}_2, s_2; n\rangle \, , |\vec{p}_i, s_i; n\rangle \in \mathscr{H}_1^n \rangle \ . \tag{14.1}$$

As first and second particle does not have an invariant meaning, we introduce subspace such that $|\vec{p}_1, s_1, n, \vec{p}_2, s_2, n\rangle$ and $|\vec{p}_2, s_2, n, \vec{p}_1, s_1, n\rangle$ are physically equivalent, which means that

$$|\vec{p}_1, s_1, n, \vec{p}_2, s_2, n\rangle \stackrel{!}{=} \alpha \, |\vec{p}_2, s_2, n, \vec{p}_1, s_1, n\rangle \tag{14.2}$$

with a phase. There comes to the next question: what α depends on?

14.1.1 Spin/helicity dependence of α

The subspace we introduce must carry a representation of the Poincaré group. First, consider two particles of fiducial momentum k, the Hilbert space

$$V_1 \otimes V_1 = (V_1 \otimes V_1)_s \oplus (V_1 \otimes V_1)_a \, , \tag{14.3}$$

where $V_1 = \langle |k,s,n\rangle\rangle$ and the subscript s means the symmetrized operation while a means the anti-symmetrized operation,

$$(V_1 \otimes V_1)_{s,a} = \left\langle \frac{1}{\sqrt{2}} \left(|k,s_1,n\rangle \otimes |k,s_2,n\rangle \pm |k,s_2,n\rangle \otimes |k,s_1,n\rangle\right)\right\rangle , \tag{14.4}$$

then $\alpha = \pm 1$ because of the irreduciblity (can only lives in one subspace), so it is independent of spin/helicity.

14.1.2 Momentum dependence of α

If α depends on momentum, it must depend on p_1, p_2. By Lorentz invariance, it should

$$|\vec{p}_1, s_1, n; \vec{p}_2, s_2, n\rangle = \alpha(p_1 \cdot p_2)\alpha(p_2 \cdot p_1)\, |\vec{p}_1, s_1, n; \vec{p}_2, s_2, n\rangle \,, \Rightarrow \quad \alpha^2(p_1 \cdot p_2) = 1 \,, \quad \alpha(p_1 \cdot p_2) = \pm 1 \,. \tag{14.5}$$

If we impose continuous dependence on momentum, the only possible situation is that it is momentum independent.

Definition 14.2. *If $\alpha(n) = 1$, n is a boson. $\alpha(n) = -1$, n is a fermion.*

14.1.3 Anyons

In two dimensional case, there exists much more particle types (α can take arbitrary $U(1)$ value), which is called the anyon.

14.2 The spin-statistic theorem

Bosons transform in integer spin representaion of Lorentz group (i.e. irreps of little group $SU(2)$, labelled by $\mathbb{Z}$). Fermions transform in half-integer spin representations. We will check this argument via the Lorentz invariance of time-ordering. For time-ordering to be a Lorentz invariant concept, the ordering of space-like separated fields should not matter.

Remark 14.1. *When $|x-y|^2 < 0$, $sgb(x^0 - y^0)$ is Lorentz frame dependent.*

Following this spirit, we study the (anti) commutation relations of space-like separated fields.

14.2.1 Real scalar field

For real scalar field

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p^\dagger e^{ipx} + a_p e^{-ipx}\right) , \tag{14.6}$$

$$
\begin{aligned}
[\phi(x),\phi(y)] &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\left([a_p^\dagger,a_q]e^{ipx-iqy}+[a_p,a_q^\dagger]e^{-ipx+iqy}\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(-e^{-ip(x-y)}+e^{ip(y-x)}\right)\\
&= \int \frac{d^4p}{(2\pi)^4}\delta(p^2-m^2)\Theta(p^0)\left(-e^{-ip(x-y)}+e^{ip(y-x)}\right)\\
&= \int \frac{d^4\Lambda p}{(2\pi)^4}\delta(p^2-m^2)\Theta(p^0)\left(-e^{-i(\Lambda^{-1}p,x-y)}+e^{-i(\Lambda^{-1}p,x-y)}\right)\\
&\overset{\det\Lambda=1}{=} \int \frac{d^4\Lambda p}{(2\pi)^4}\delta(p^2-m^2)\Theta(p^0)\left(-e^{i(p,\Lambda(x-y))}+e^{-i(p,\Lambda(x-y))}\right)\,.
\end{aligned}
\tag{14.7}
$$

If $(x-y)^2<0$, we can choose Λ such that $(\Lambda(x-y))^0=0$, which gives

$$
\Rightarrow \quad [\phi(x),\phi(y)] \overset{(x-y)^2<0}{=} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}+e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\right)=0\,. \tag{14.8}
$$

14.2.2 Dirac field

$$
\psi(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{\sigma=\pm1}\left[u_\sigma(p)a_\sigma(p)e^{-ipx}+v_\sigma(p)b_\sigma^\dagger(p)e^{ipx}\right]\,, \tag{14.9}
$$

Choose canonical commutation of anti commutation relations? We denote the anticommutator as $[\cdot,\cdot]_+$ and continue the calculation to see the result.

$$
\begin{aligned}
[\psi(x),\psi^\dagger(y)]_\pm = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\sum_{\sigma\sigma'}\Big(&e^{-ipx+iqy}u_\sigma(p)u_{\sigma'}^\dagger(p)[a_\sigma(p),a_{\sigma'}]\pm\\
&+e^{ipx-iqy}v_\sigma(p)v_{\sigma'}^\dagger(q)[b_\sigma^\dagger(p),b_{\sigma'}(q)]_\pm\Big)\,.
\end{aligned}
\tag{14.10}
$$

We set

$$
\begin{aligned}
[a_\sigma(p),a_{\sigma'}]_\pm &= (2\pi)^3\delta_{\sigma\sigma'}\delta^{(3)}(\vec{p}-\vec{q})\\
[b_\sigma^\dagger(p),b_{\sigma'}(q)]_\pm &= \pm(2\pi)^3\delta_{\sigma\sigma'}\delta^{(3)}(\vec{p}-\vec{q})\,,
\end{aligned}
\tag{14.11}
$$

where we notice that what important is the relative sign, so we can fix the result at the first line. We will determine the sign (commutator or anti-commutator) later. We also need to finish the spin sum,

$$
\sum_\sigma u_\sigma(p)u_\sigma^\dagger(p)=\sum_\sigma L(p)_S u_\sigma(k)u_\sigma^\dagger(k)L(p)_S^\dagger\,, \tag{14.12}
$$

with

$$
\begin{aligned}
\sum_\sigma u_\sigma(k)u_\sigma^\dagger(k) &= \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+\sqrt{m}\begin{pmatrix}1&0&1&0\end{pmatrix}+\sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\sqrt{m}\begin{pmatrix}0&1&0&1\end{pmatrix}\\
&= m\begin{pmatrix}\mathbb{1}&\mathbb{1}\\\mathbb{1}&\mathbb{1}\end{pmatrix}=m\left(\mathbb{1}_{4\times4}+\gamma^0\right)\,,
\end{aligned}
\tag{14.13}
$$

the spin sum reduces to

$$
\sum_\sigma u_\sigma(p)u_\sigma^\dagger(p)=m\left(L(p)_S\gamma^0L(p)_S^{-1}\gamma^0+\gamma^0\right)\,, \tag{14.14}
$$

where we used $(\gamma^0)^2 = \mathbb{1}$. With $\Lambda^{-1}\gamma^\mu\Lambda = \Lambda^\mu{}_\nu\gamma^\nu$,

$$L(p)_S\gamma^0 L(p)_S^{-1} = L^{-1}(p)^0{}_\mu\gamma^\mu\,, \tag{14.15}$$

with $L^{-1}(p)^0{}_\mu = \frac{p_\mu}{m}$, we have

$$\sum_\sigma u_\sigma(p)u_\sigma^\dagger(p) = (\not{p}+m)\gamma^0\,. \tag{14.16}$$

Likewise

$$\sum_\sigma v_\sigma(p)v_\sigma^\dagger(p) = L(p)_S\sum_\sigma v_\sigma(k)v_\sigma^\dagger(k)L^\dagger(p)_S = (\not{p}-m)\gamma^0\,, \tag{14.17}$$

as $\sum_\sigma v_\sigma(k)v_\sigma^\dagger(k) = m(\mathbb{1}-\gamma^0)$. Then

$$\begin{aligned}[]
[\psi(x),\psi^\dagger(y)]_\pm &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-ip(x-y)}(\not{p}+m)\pm e^{ip(x-y)}(\not{p}-m)\right)\gamma^0 \\
= (i\not{\partial}+m)[\psi(x),\psi^\dagger(y)]_\pm &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{-ip(x-y)}\mp e^{ip(x-y)}\right)\gamma^0\,,
\end{aligned} \tag{14.18}$$

In order to let result vanishes when the fields are space-like separated, we need to anticommutator $[\cdot,\cdot]_+$.

Remark 14.2. *Recall the specific form of $u(k)$ and $v(k)$. It is good under the parity and time reversal transformation.*

We need to impose

$$\{a_\sigma(p),a^\dagger_{\sigma'}(q)\} = [a_\sigma(p),a^\dagger_{\sigma'}(q)]_+ = (2\pi)^3\delta_{\sigma\sigma'}\delta^{(3)}(\vec{p}-\vec{q})\,, \tag{14.19}$$

and likewise for anti particles. We then extend to equal time canonical anti-commutator relations

$$\{a_\sigma(p),a_{\sigma'}(q)\} = \{a^\dagger_\sigma(p),a^\dagger_{\sigma'}(q)\} = 0 = \{a_\sigma(p),b_{\sigma'}(q)\} = \dots \tag{14.20}$$

It gives,

$$\{\psi(x),\psi(y)\} = 0\,, \tag{14.21}$$

which forces a modification of the definition of the time-ordering of fermions. For $(x-y)^2<0$

$$\mathscr{T}\left\{\psi(x)\psi^\dagger(y)\right\} = \mathscr{T}\left\{-\psi^\dagger(y)\psi(x)\right\} = -\mathscr{T}\left\{\psi^\dagger(y)\psi(x)\right\}\,, \tag{14.22}$$

where the last equal comes from the linearity of time order product. If we denote $\Psi = \psi,\psi^\dagger$

$$\Rightarrow\quad \mathscr{T}\left\{\Psi(x_1),\dots,\Psi(x_n)\right\} = \mathrm{sgn}(\sigma)\Psi(x_{\sigma(1)})\dots\Psi(x_{\sigma(m)})\,, \tag{14.23}$$

where $x^0_{\sigma(1)} > \dots x^0_{\sigma(n)}$, $\mathrm{sgn}(\sigma) = (-1)^{\#\text{ of transposition to get }\sigma}$.

Example 14.1.

$$T\{\psi(x)\psi(y)\} = \Theta(x^0-y^0)\psi(x)\psi(y) - \Theta(y^0-x^0)\psi(y)\psi(x)\,. \tag{14.24}$$

14.3 Time ordering for interacting fields

Time ordering prescription is Lorentz invariant. For interacting fields, it becomes

$$\langle\Omega|\,\mathscr{T}\left\{\Phi(x_1)\ldots\Phi(x_n)\right\}|\Omega\rangle = \frac{\langle 0|\,\mathscr{T}\left\{\Phi_0(x_1)\ldots\Phi_0(x_n)e^{i\int\mathscr{L}_{\mathrm{int}}d^4x}\right\}|0\rangle}{\langle 0|\,\mathscr{T}\left\{e^{i\int\mathscr{L}_{\mathrm{int}}d^4x}\right\}|0\rangle}\,. \tag{14.25}$$

We need to extend the relation to matrix element between arbitrary basis vectors, it is straightforward for choice of in/out states as basis.

Chapter 15

Quantum Electrodynamics

The Lagrangian we focus on is the QED Lagrangian with the minimal coupling to photon

$$\mathscr{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \quad \text{with} \quad D_\mu = \partial_\mu + ieA_\mu\,. \tag{15.1}$$

15.1 The Dirac propagator

We want to first compute

$$\langle 0|\,\mathscr{T}\left\{\psi_\alpha(0)\bar{\psi}_\beta(x)\right\}|0\rangle\ , \tag{15.2}$$

where α, β are spinor indices. The vacuum expectation value is

$$\begin{aligned}
\langle 0|\,\psi_\alpha(0)\bar{\psi}_\beta(x)\,|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{d^3q}{(2\pi)^3}\frac{e^{iqx}}{\sqrt{2\omega_q}}\sum_{\sigma\sigma'}u^\alpha_\sigma(p)\bar{u}^\beta_{\sigma'}(q)\,\langle 0|\,a_\sigma(p)a^\dagger_{\sigma'}(q)\,|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{d^3q}{(2\pi)^3}\frac{e^{iqx}}{\sqrt{2\omega_q}}\sum_{\sigma\sigma'}u^\alpha_\sigma(p)\bar{u}^\beta_{\sigma'}(q)\,\langle 0|\,\{a_\sigma(p), a^\dagger_{\sigma'}(q)\}\,|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{d^3q}{(2\pi)^3}\frac{e^{iqx}}{\sqrt{2\omega_q}}\sum_{\sigma\sigma'}u^\alpha_\sigma(p)\bar{u}^\beta_{\sigma'}(q)(2\pi)^3\delta_{\sigma\sigma'}\delta^{(3)}(\vec{p}-\vec{q}) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{e^{ipx}}{2\omega_p}\sum_\sigma u^\alpha_\sigma(p)\bar{u}^\beta_\sigma(p)\,,
\end{aligned} \tag{15.3}$$

Notice that we finish the spin sum before,

$$\sum_\sigma u^\alpha_\sigma(p)\bar{u}^\beta_\sigma(p) = (\not{p} + m)_{\alpha\beta}\,, \tag{15.4}$$

we have

$$\langle 0|\,\psi_\alpha(0)\bar{\psi}_\beta(x)\,|0\rangle = (-i\not{\partial} + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{ipx}\,. \tag{15.5}$$

Similarly,

$$\begin{aligned}
\langle 0|\,\psi_\beta(x)\bar{\psi}_\alpha(0)\,|0\rangle &= \int \frac{d^3p}{\sqrt{2\omega_p}}\frac{d^3q}{\sqrt{2\omega_q}}e^{-ipx}\sum_{\sigma\sigma'}\bar{v}^\beta_\sigma(p)v^\alpha_{\sigma'}(q)\,\langle 0|\,b_\sigma(p)b^\dagger_{\sigma'}(q)\,|0\rangle \\
&= -(-i\not{\partial} + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}e^{-ipx}\,,
\end{aligned} \tag{15.6}$$

then the time order product is

$$\begin{aligned}\langle 0|\mathscr{T}\left\{\psi(0)\bar{\psi}(x)\right\}|0\rangle &= \langle 0|\psi(0)\bar{\psi}(x)|0\rangle\,\Theta(-t) - \langle 0|\bar{\psi}(x)\psi(0)|0\rangle\,\Theta(t) \\ &= (-i\not{\partial}+m)\int\frac{d^3p}{(2\pi)^3}\frac{e^{ipx}\Theta(-t)+e^{-ipx}\Theta(t)}{2\omega_p}\,.\end{aligned} \tag{15.7}$$

Here the ∂_t will act on the Heaviside function, adding δ function term. However, it is cancelled with each other so we can safely move the derivative out of the whole term. Using the formula (5.8), we write

$$\begin{aligned}\langle 0|\mathscr{T}\left\{\psi(0)\bar{\psi}(x)\right\}|0\rangle &= (-i\not{\partial}+m)\int\frac{d^4p}{(2\pi)^4}\frac{ie^{ipx}}{(p^0)^2-\omega_p^2+i\epsilon} \\ &= \int\frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}e^{ipx}\,.\end{aligned} \tag{15.8}$$

By translation invariance of the vacuum, i.e. $e^{ipx}|0\rangle = |0\rangle$, we have

$$\begin{aligned}\langle 0|\mathscr{T}\left\{\psi(y)\bar{\psi}(x)\right\}|0\rangle &= (-i\not{\partial}+m)\int\frac{d^4p}{(2\pi)^4}\frac{ie^{ipx}}{(p^0)^2-\omega_p^2+i\epsilon} \\ &= \int\frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}e^{ip(x-y)}\,.\end{aligned} \tag{15.9}$$

The graphical description is

$$\bar{\psi}_\beta(x) \longrightarrow \psi_\alpha(y) \tag{15.10}$$

15.2 LSZ and external propagator

The LSZ reduction formula of QED is

$$\begin{aligned} i\int d^4x e^{ipx}(\Box+m^2)\psi(x) &= \sqrt{2\omega_p}\sum_{\sigma=\pm\frac{1}{2}}u_\sigma(p)\left(a_{p,\sigma}(\infty)-a_{p,\sigma}(-\infty)\right) \\ i\int d^4x e^{-ipx}(\Box+m^2)\psi(x) &= \sqrt{2\omega_p}\sum_{\sigma=\pm\frac{1}{2}}v_\sigma(p)\left(b^\dagger_{p,\sigma}(-\infty)-b^\dagger_{p,\sigma}(\infty)\right) \\ i\int d^4x e^{ipx}(\Box+m^2)\bar{\psi}(x) &= \sqrt{2\omega_p}\sum_{\sigma=\pm\frac{1}{2}}\bar{v}_\sigma(p)\left(b_{p,\sigma}(\infty)-b_{p,\sigma}(-\infty)\right) \\ i\int d^4x e^{-ipx}(\Box+m^2)\bar{\psi}(x) &= \sqrt{2\omega_p}\sum_{\sigma=\pm\frac{1}{2}}\bar{u}_\sigma(p)\left(a^\dagger_{p,\sigma}(-\infty)-a^\dagger_{p,\sigma}(\infty)\right)\,.\end{aligned} \tag{15.11}$$

Pay attention that the plus sign of e^{ipx} means the outgoing particle while the minus sign means the incoming particle. We need to project the field onto state of given spins, by using

$$\bar{u}_\sigma(p)u_{\sigma'}(p) = 2m\delta_{\sigma\sigma'} = -\,\delta v_\sigma\,(p)v_{\sigma'}(p)\,, \tag{15.12}$$

the summation will select the given spin we want, then we have

- For incoming particle $(a^\dagger(-\infty))$, insert $\bar{\psi}(x)u_\sigma(p)/2m$.

- For incoming anti-particle ($b^\dagger(-\infty)$), insert $-\bar{v}_\sigma(p)\psi(x)/2m$.
- For outgoing particle ($a(\infty)$), insert $\bar{u}_\sigma(p)\psi(x)/2m$.
- For outgoing anti-particle ($b(\infty)$), insert $-\bar{\psi}(x)v_\sigma(p)/2m$.

Example 15.1. *For concreteness we check one example to see what happen if we do the insertion. For instance, we check the effect of $\bar{\psi}(x)u_\sigma(p)/2m$ insertion (incoming particle),*

$$\begin{aligned}
&i\int d^4x e^{-ipx}(\Box + m^2)\bar{\psi}(x)\frac{1}{2m}u_\sigma(p) \\
\longrightarrow\quad & i\int d^4x e^{-ipx}(-p^2+m^2)\int\frac{d^4q}{(2\pi)^4}\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon}e^{iq(x-y)}\frac{1}{2m}u_\sigma(q) \\
&= ie^{-ipy}(-p^2+m^2)\frac{i(\not{p}+m)}{p^2-m^2}\frac{1}{2m}u_\sigma(p) \\
&= e^{-ipy}\frac{1}{2m}(\not{p}+m)u_\sigma(p) \\
&= e^{-ipy}u_\sigma(p)
\end{aligned} \tag{15.13}$$

where we evaluate this expression with ψ and their contractions, also the $(\not{p}-m)u_\sigma(p)=0$. We set the incoming momentum is on-shell. The propagator is amputated and replaced by polarization tensor.

Analogous computations for the other three cases yield the effect of insertions.

- Place external momentum on-shell
- amputate propagator, replace by appropriate Dirac polarization tensor

15.3 Momentum space Feynman rules

15.3.1 Propagators

- The photon propagator

$$\mu \overset{\xrightarrow{p}}{\rightsquigarrow} \nu \;=\; \frac{-i}{p^2+i\epsilon}\left[g_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}\right]. \tag{15.14}$$

- The Fermion propagator

$$\bar{\psi}_\beta(x) \overset{\xrightarrow{p}}{\blacktriangleright} \psi_\alpha(y) \;=\; \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}. \tag{15.15}$$

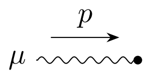
15.3.2 External lines

Again, the dot means the external point (the graph or the flow ends at this point).

- Incoming photon

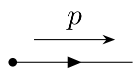
$$\bullet\overset{\xrightarrow{p}}{\rightsquigarrow} \mu \;=\varepsilon_\mu(p)\,. \tag{15.16}$$

- Outgoing photon



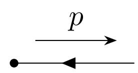
$$= \varepsilon^*_\mu(p)\,. \tag{15.17}$$

- Incoming particle (fermion)



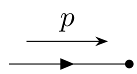
$$= u_\sigma(p)\,. \tag{15.18}$$

- Incoming antiparticle (fermion)



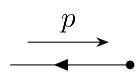
$$= \bar{v}_\sigma(p)\,. \tag{15.19}$$

- Outgoing particle (fermion)



$$= \bar{u}_\sigma(p)\,. \tag{15.20}$$

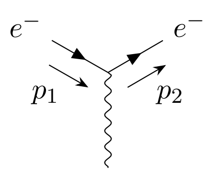
- Outgoing antiparticle (fermion)



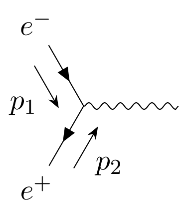
$$= v_\sigma(p)\,. \tag{15.21}$$

15.3.3 Vertices

The corresponding interaction Lagrangian is $\mathscr{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu$, we have the trivalent vertex taking the value



$=$

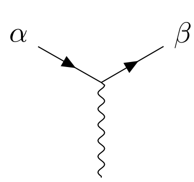


$$= \ldots = -ie\gamma^\mu\,. \tag{15.22}$$

15.3.4 Index contractions

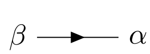
- μ of γ^μ contracts with
  - $g^{\mu\nu}$ and the gauge term of the photon propagator.
  - polarization tensor $\varepsilon^\mu, \varepsilon^{\mu *}$ of external photons.
- The spinor indices, which can be made explicit by writing $\mathscr{L}_{\text{int}} = -e\bar{\psi}_\alpha\gamma^\mu_{\alpha\beta}\psi_\beta A_\mu$, generates the graph

  $\Rightarrow$

  $$= -ie\gamma^\mu_{\alpha\beta}\,, \tag{15.23}$$

  contracts with
  - Dirac propagator

    $$= \frac{i(\not{p}+m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}\,. \tag{15.24}$$

  - external spinors.

15.3.5 Sign I

We need to be careful about the relative signs due to

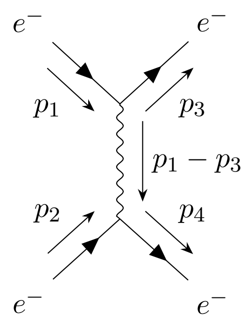
$$\mathscr{T}\left\{\Psi(x)\Psi(y)\right\} = -\mathscr{T}\left\{\Psi(y)\Psi(x)\right\} . \tag{15.25}$$

To be safe, we start with some ordering or fields inside $\mathscr{T}$, then reorder keeping track of signs to have the fermionic fields that are contracted next to each other, in the order $\Psi(x)\bar{\Psi}(y)$.

15.3.6 Example: Møller scattering

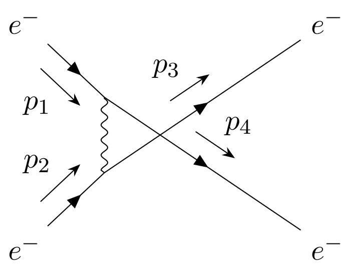
Møller scattering: $e^-e^- \to e^-e^-$. We have the following two Feynman diagrams,

$i\mathscr{M}_t =$



$$= (-ie)^2\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma^\nu u(p_2)\frac{-ig_{\mu\nu}}{(p_1-p_3)^2+i\epsilon} , \tag{15.26}$$

$i\mathscr{M}_u =$



$$= -(-ie)^2\bar{u}(p_4)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma^\nu u(p_2)\frac{-ig_{\mu\nu}}{(p_1-p_4)^2} , \tag{15.27}$$

where we temporarily hide the spin label σ on spinors and pay attention the red minus sign comes from the anti-commutation of the Dirac field. Now we explicitly check the minus relative sign. Choose a random order

$$\begin{aligned} &\langle\Omega|\,\mathscr{T}\left\{\psi_{\alpha_3}(x_3)\bar{\psi}_{\alpha_1}(x_1)\psi_{\alpha_4}(x_4)\bar{\psi}_{\alpha_2}(x_2)\right\}|\Omega\rangle \\ \to\quad &\langle 0|\,\mathscr{T}\left\{\psi_{\alpha_3}(x_3)\bar{\psi}_{\alpha_1}(x_1)\psi_{\alpha_4}(x_4)\bar{\psi}_{\alpha_2}(x_2)(-ie)\int d^4x\bar{\psi}(x)\not{A}\psi(x)\int d^4y\bar{\psi}(y)\not{A}\psi(y)\right\}|0\rangle \\ &= (-ie)^2\gamma^\mu_{\beta_1\beta_2}\gamma^\mu_{\beta_3\beta_4}\int d^4x\int d^4y \\ &\langle 0|\,\mathscr{T}\left\{\psi_{\alpha_3}(x_3)\bar{\psi}_{\alpha_1}(x_1)\psi_{\alpha_4}(x_4)\bar{\psi}_{\alpha_2}(x_2)\bar{\psi}(x)_{\beta_1}\psi(x)_{\beta_2}\bar{\psi}(y)_{\beta_3}\psi(y)_{\beta_4}A_\mu(x)A_\nu(y)\right\}|0\rangle , \end{aligned} \tag{15.28}$$

For t-channel (15.26), the contraction align would be

$$\psi_{\alpha_3}(x_3)\bar{\psi}_{\beta_1}(x)\psi_{\beta_2}(x)\bar{\psi}_{\alpha_1}(x_1)\psi_{\alpha_4}(x_4)\bar{\psi}(y)_{\beta_3}\psi(y)_{\beta_4}\bar{\psi}(x_2)_{\alpha_2} , \tag{15.29}$$

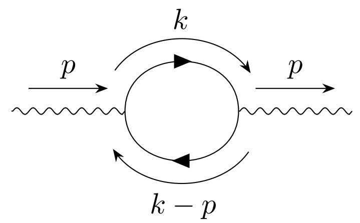
with no extra sign ($\bar\psi_{\beta_i}\psi_{\beta_j}$ moves as a block). For u-channel (15.27), the contraction is

$$-\psi_{\alpha_4}(x_4)\bar\psi_{\beta_1}(x)\psi_{\beta_2}(x)\bar\psi_{\alpha_1}(x_1)\psi_{\alpha_3}(x_3)\bar\psi(y)_{\beta_3}\psi(y)_{\beta_4}\bar\psi(x_2)_{\alpha_2} \tag{15.30}$$

is indeed with a extra sign.

15.3.7 Signs for fermion loops

A $\bar\psi\gamma^\mu\psi$ fermion pair is separated when evaluating fermion loops.



(15.31)

The contraction looks like

$$\begin{aligned}(-ie)^2\gamma^\mu_{\alpha_1\alpha_2}\gamma^\nu_{\beta_1\beta_2}\langle 0|\,\mathscr{T}\left\{A_\mu(x)A_\nu(y)\bar\psi_{\alpha_1}(x)\psi_{\alpha_2}(x)\bar\psi_{\beta_1}(y)\psi_{\beta_2}(y)\right\}|0\rangle \\ \to \quad -\psi_{\beta_2}(y)\bar\psi_{\alpha_1}(x)\psi_{\alpha_2}(x)\bar\psi_{\beta_1}(y)\,.\end{aligned} \tag{15.32}$$

15.3.8 Sign II

Sign rules: a relative sign appears

- For each exchange of external fermions.
- For each fermion loop.

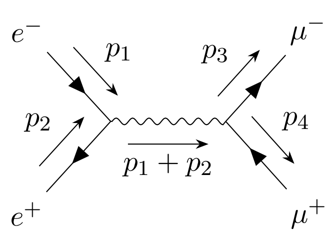
15.4 $e^+e^-\to\mu^+\mu^-$ scattering (and spin sums)

Any electrically charged fermion contributes to the Lagrangian via

$$\bar\psi_n(i\not{D}_n - m_n)\psi_n\,,\quad \not{D}_n = \gamma^\mu(\partial_\mu - ieQ_nA_\mu)\,. \tag{15.33}$$

In this scattering process, $Q_e=-1$, $m_e=511\text{KeV}$, which is the electron. $Q_\mu=-1$, $m_\mu=106\text{MeV}$, which is the muon. The Feynman diagram is

$i\mathscr{M} =$



$$= (-ie)^2\bar v_e(p_2)\gamma^\rho u_e(p_1)\bar u_\mu(p_3)\gamma^\nu v_\mu(p_4)\frac{-i\left(g_{\rho\nu}-(1-\xi)\frac{k_\rho k_\nu}{m^2}\right)}{(p_1+p_2)^2}\,, \tag{15.34}$$

the ξ-dependence can be extracted as

$$\bar{v}_e(p_2)(\not{p}_1 + \not{p}_2)u_e(p_1) = \bar{v}_e(p_2)(m_e - m_e)u_e(p_1) = 0\,, \tag{15.35}$$

where we use $(\not{p} - m)u(p) = \bar{v}(p)(\not{p} + m) = 0$. Likewise for the second spinor bilinear, ξ-dependence vanishes. To obtain the cross-section, need $|\mathscr{M}|^2 = \mathscr{M}^\dagger \mathscr{M}$,

$$(\bar{v}_e(p_2)\gamma^\mu u_e(p_1))^\dagger = u_e(p_1)^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger v_e(p_2) = u_e(p_1)^\dagger\gamma^0\gamma^\mu v_e(p_2) = \bar{u}_e(p_1)\gamma^\mu v_e(p_2)\,, \tag{15.36}$$

the amplitude is then

$$|\mathscr{M}|^2 = \frac{e^4}{s^2}\left[\bar{v}_e(p_2)\gamma^\rho u_e(p_1)\bar{u}_\mu(p_3)\gamma_\rho v_\mu(p_4)\right]\left[\bar{u}_e(p_1)\gamma^\nu v_e(p_2)\,\delta v_\mu\,(p_4)\gamma_\nu u_\mu(p_3)\right]\,. \tag{15.37}$$

Unpolarized Scattering

If we do not measure spin in outgoing channel, we should sum over all outgoing spins,

$$\sum_{\sigma_3,\sigma_4} |\mathscr{M}(\sigma_1,\sigma_2,\sigma_3,\sigma_4)|^2 = |\mathscr{M}|^2_{\text{unpolarized}}\,. \tag{15.38}$$

All the polarization related to muon should be sum over,

$$\begin{aligned} |\mathscr{M}|^2_{\text{unpolarized}} &= \frac{e^4}{s^2}\bar{v}_e(p_2)\gamma^\rho u_e(p_1)\bar{u}_e(p_1)\gamma^\nu v_e(p_2) \\ &\times \sum_{\sigma_3\sigma_4} \bar{u}^{\sigma_3}_\alpha(\gamma_\rho)_{\alpha\beta}v^{\sigma_4}_\beta\bar{v}^{\sigma_4}_\gamma(\gamma_\nu)_{\gamma\delta}u^{\sigma_3}_\delta\,, \end{aligned} \tag{15.39}$$

with

$$\sum_\sigma u^\sigma(p_3)\bar{u}^\sigma(p_3) = \not{p}_3 + m\,, \quad \sum_\sigma v^\sigma(p_4)\bar{v}^\sigma(p_4) = \not{p}_4 - m\,. \tag{15.40}$$

the summation reduces to

$$\begin{aligned} &\times \sum_{\sigma_3\sigma_4} \bar{u}^{\sigma_3}_\alpha(\gamma_\rho)_{\alpha\beta}v^{\sigma_4}_\beta\bar{v}^{\sigma_4}_\gamma(\gamma_\nu)_{\gamma\delta}u^{\sigma_3}_\delta \\ &= (\not{p}_3 + m)_{\delta\alpha}(\gamma_\rho)_{\alpha\beta}(\not{p}_4 - m)_{\beta\gamma}(\gamma_\nu)_{\delta\gamma} = \text{Tr}\left[(\not{p}_3 + m_\mu)\gamma_\rho(\not{p}_4 - m_\mu)\gamma_\nu\right]\,. \end{aligned} \tag{15.41}$$

Averaging over incoming spinors

If incoming beams are unpolarized: density matrix with equal probability 1/2 for either spin, we need to sum over the spins, weighted by $(1/2)^2$. Thus the result is

$$\begin{aligned} \frac{1}{4}\sum_{\text{spins}} |\mathscr{M}|^2 &= \frac{e^4}{s^2}\,\text{Tr}\left[(\not{p}_1 + m_e)\gamma^\nu(\not{p}_2 - m_e)\gamma^\rho\right]\text{Tr}\left[(\not{p}_3 + m_\mu)\gamma_\rho(\not{p}_4 - m_\mu)\gamma_\nu\right] \\ &= \frac{2e^4}{s^2}\left[u^2 + t^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2\right]\,. \end{aligned} \tag{15.42}$$

The differential cross section in center mass frame for $2 \to 2$ scattering is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{1}{64\pi^2 E_{\text{CM}}^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathscr{M}|^2 \,. \tag{15.43}$$

In ultra-relativistic limit $m_e, m_\mu \ll E$,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\alpha^2}{4E_{\text{CM}}^2} (1 + \cos^2\theta) \,. \tag{15.44}$$

Chapter 16

Path Integral

16.1 Warm up: quantum mechanics in 1D system via path integral

Typically $H(\hat{x},\hat{p}) = \frac{\hat{p}^2}{2m} + V(x)$, with canonical commutation relation $[\hat{x},\hat{p}] = i$. For more general H, we choose to order operators with $\hat{p}$ to the left of $\hat{x}$. In Heisenberg picture

$$\begin{aligned} \hat{x}(t) &= e^{i\hat{H}t}\hat{x}e^{-i\hat{H}t}, \\ \hat{p}(t) &= e^{i\hat{H}t}\hat{p}e^{-i\hat{H}t} \end{aligned} \tag{16.1}$$

states are time-independent, but the eigenstates are time-independent, the corresponding operators are

$$\hat{x}\,|x\rangle = x\,|x\rangle \Rightarrow \hat{x}(t)\,|x(t)\rangle = \hat{x}(t)e^{i\hat{H}t}\,|x\rangle = e^{i\hat{H}t}x\,|x\rangle = x\,|x,t\rangle \ . \tag{16.2}$$

$|x,t\rangle$ is the eigenstate of the eigenvalue x of operator $\hat{x}(t)$. The task (with application to QFT) is that for given two eigenstates $|x\rangle$, $|x'\rangle$ of $\hat{x}$, evaluate $\langle x', t_f | x, t_i\rangle$.

$$\left\langle x', t_f \middle| x, t_i \right\rangle = \left\langle x' \middle| e^{i\hat{H}(t_f - t_i)} \middle| x \right\rangle \ , \tag{16.3}$$

First, assume $\Delta t = t_f - t_i \ll 1$, we can use the linear approximation of $e^{-iH\Delta t}$,

$$\left\langle x', t_i + \Delta t \middle| x, t_i \right\rangle = \left\langle x' \middle| \mathbb{1} - iH(\hat{x},\hat{p})\Delta t \middle| x \right\rangle = \left\langle x' \middle| e^{-iH(x,\hat{p})\Delta t} \middle| x \right\rangle \ . \tag{16.4}$$

Introducing $\mathbb{1} = \int |p\rangle\langle p|$, we have

$$\begin{aligned} &= \int dp \left\langle x' \middle| p \right\rangle \langle p|\, e^{-iH(x,\hat{p})\Delta t}\,|x\rangle \\ &= \int dp \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{ipx' - ipx} e^{-iH(x,p)\Delta t} \\ &= \frac{1}{2\pi} e^{-iV(x)\Delta t} \int dp e^{-i\frac{p^2}{2m}\Delta t + ip(x'-x)} \\ &= \sqrt{\frac{2m}{\Delta t}}\frac{1}{2\sqrt{\pi}} e^{i\Delta t\left[\frac{m}{2}\left(\frac{x'-x}{\Delta t}\right)^2 - V(x)\right]}, \end{aligned} \tag{16.5}$$

where the factor in the exponential is the Lagrangian $L\left(x, \frac{x'-x}{\Delta t}\right)$. For finite time $t_f - t_i$, the strategy is to divide the internal into $(n+1)$ pieces: $t_f - t_i = (n+1)\Delta t$. Introduce at each $t_k = t_i + k\Delta t$,

$$\mathbb{1} = \int dx \left|x\right\rangle \left\langle x\right| = e^{i\hat{H}t_k} \int dx \left|x\right\rangle \left\langle x\right| e^{-i\hat{H}t_k} = \int dx \left|x, t_k\right\rangle \left\langle x, t_k\right| . \tag{16.6}$$

$$\begin{aligned} \Rightarrow \quad \left\langle x', t_f \middle| x, t_i \right\rangle &= \int dx_1 \dots dx_n \left\langle x', t_f \middle| x_n, t_n \right\rangle \left\langle x_n, t_n \middle| x_{n-1}, t_{n-1} \right\rangle \dots \left\langle x_1, t_1 \middle| x_i, t_i \right\rangle \\ &= N^{n+1} \int dx_1 \dots dx_n e^{i\Delta t\left[L\left(x_n, \frac{x'-x_n}{\Delta t} + \dots L\left(x_i, \frac{x_1 - x_i}{\Delta t}\right)\right)\right]} , \end{aligned} \tag{16.7}$$

where $N = \sqrt{\frac{2m}{\Delta t}\frac{1}{2\sqrt{\pi}}}$. String together $x_1, \dots, x_n$ into differentiable function[1] $x(t)$ with $x(t_i) = x$, $x(t_f) = x'$. Define a measure $\mathcal{D}(x(t))$ on this space of differentiable function

$$\mathcal{D}(x(t)) = \lim_{n\to\infty} \prod_{i=1}^{n} dx_i \, , \tag{16.8}$$

such that

$$\left\langle x', t_f \middle| x, t_i \right\rangle = \tilde{N} \int_{x(t_i)=x}^{x(t_f)=x'} \mathcal{D}x(t) e^{i\int_{t_i}^{t_f} dt L(x,\dot{x})} = \tilde{N} \int_{x(t_i)=x}^{x(t_f)=x'} \mathcal{D}x(t) e^{iS} . \tag{16.9}$$

Remark 16.1. *In classical limit $\hbar \to 0$, stationary phase argument applies to $e^{iS/\hbar}$. Path integral is dominated by stationary phase of S, i.e. by classical trajectory (solution of EoM).*

16.2 Path integral in QFT (bosonic)

16.2.1 Completeness relation in QFT

One particle quantum mechanics: the eigenbasis of $\hat{x}$ is $\hat{x}\left|x\right\rangle = x\left|x\right\rangle$. In the n particle quantum mechanics, the eigenbasis of $\hat{x}_i$ is

$$\hat{x}_i \left|x_1, \dots, x_n\right\rangle = x_i \left|x_1, \dots, x_n\right\rangle , \tag{16.10}$$

with $[\hat{x}_i, \hat{x}_j] = 0$. The QFT eigenbasis of $\hat{\phi}(x)$ (where the continuous index plays the role of i),

$$\hat{\phi}(x_0) \left|\{\phi(\vec{x})\}\right\rangle = \phi(\vec{x}_0) \left|\{\phi(\vec{x})\}\right\rangle . \tag{16.11}$$

The completeness relation expressed via functional integral

$$\mathbb{1} = \int \mathcal{D}\phi(x) \left|\{\phi(x)\}\right\rangle \left\langle\{\phi(x)\}\right| . \tag{16.12}$$

Likewise, $\hat{\pi}(\vec{x}_0)\left|\{\pi(x)\}\right\rangle = \pi(\vec{x}_0)\left|\{\pi(\vec{x})\}\right\rangle$. By

$$[\hat{\phi}(x), \hat{\pi}(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \, , \tag{16.13}$$

$$\left\langle\{\pi(\vec{x})\}\middle|\{\phi(\vec{x})\}\right\rangle = \exp\left[-i\int d^3x \pi(x)\phi(x)\right] . \tag{16.14}$$

[1] The mathematical process here is not rigorous, which might be the root reason of the divergence of path integral. Too many curves are added into the integration.

Also, $\hat{\phi}(\vec{x})(t) = \hat{\phi}(\vec{x},t) = e^{i\hat{H}t}\hat{\phi}(\vec{x})e^{-i\hat{H}t}$, $|\{\phi(\vec{x})\},t\rangle = e^{i\hat{H}t}|\{\phi(\vec{x})\}\rangle$,

$$\Rightarrow \quad \left\langle\{\phi'(\vec{x})\},t_f\middle|\{\phi(\vec{x})\},t_i\right\rangle = N\int_{\phi(\vec{x},t_i)=\phi(\vec{x})}^{\phi(\vec{x},t_f)=\phi'(\vec{x})} \mathcal{D}\phi(\vec{x},t)e^{i\int d^4x\mathscr{L}_{\text{int}}(\phi,\partial_i\phi)}\,, \tag{16.15}$$

where

$$\mathcal{D}\phi(\vec{x},t) = \lim_{n\to\infty}\prod_{i=1}^{n}\mathcal{D}\phi_i(\vec{x})\,. \tag{16.16}$$

16.2.2 Time ordered product

If $\phi(x_j,t_j)$ is the eigenvalue of $\hat{\phi}(x_j,t_j)$. What is

$$\int \mathcal{D}\phi(\vec{x},t)e^{iS[\phi]}\phi(\vec{x}_j,t_j)\,? \tag{16.17}$$

Follow the argument reversed,

$$\int\prod_k \mathcal{D}\phi_k(\vec{x})\left\langle\{\phi'(\vec{x})\},t_f\right|e^{-i\hat{H}\Delta t}\,|\{\phi_n(\vec{x})\},t_n\rangle\ldots e^{-i\hat{H}\Delta t}\phi(\vec{x}_j,t_j)\,|\{\phi_j(\vec{x})\},t_j\rangle\ldots\langle\{\phi_1(\vec{x})\},t_1|\{\phi(\vec{x})\},t_i\rangle\ . \tag{16.18}$$

Notice that $\phi(\vec{x}_j,t_j)\,|\{\phi_j(\vec{x})\},t_j\rangle = \hat{\phi}(\vec{x}_j,t_j)\,|\{\phi_j(\vec{x})\},t_j\rangle$, and collapse all completeness relation, the final result is

$$\left\langle\{\phi'(\vec{x})\},t_f\right|\hat{\phi}(\vec{x}_j,t_j)\,|\{\phi(\vec{x})\},t_i\rangle\ . \tag{16.19}$$

Likewise,

$$\begin{gathered} N\int\mathcal{D}\phi(\vec{x},t)e^{iS[\phi]}\prod_{j=1}^{m}\phi(\vec{x}_j,t_j) = \\ \left\langle\{\phi'(\vec{x})\},t_j\right|\mathscr{T}\left\{\prod_{j=1}^{m}\hat{\phi}(\vec{x}_j,t_j)\right\}|\{\phi(\vec{x})\},t_i\rangle\,. \end{gathered} \tag{16.20}$$

The time ordering is naturally emerged!

16.2.3 Projecting onto the vacuum

By inserting the completeness relation

$$\begin{gathered} \langle\Omega|\,\mathscr{T}\left\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\right\}|\Omega\rangle = \\ \int\mathcal{D}\phi'(\vec{x})\mathcal{D}\phi(\vec{x})\left\langle\Omega\middle|\{\phi'(\vec{x})\},\infty\right\rangle\left\langle\{\phi'(\vec{x})\},\infty\right|\mathscr{T}\left\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\right\}|\{\phi(\vec{x})\},\infty\rangle\,\langle\{\phi(\vec{x})\},\infty|\Omega\rangle\ , \end{gathered} \tag{16.21}$$

Define

$$\begin{gathered} \mathscr{M}^*_{\phi'} = \left\langle\Omega\middle|\{\phi'(\vec{x})\},\infty\right\rangle\ , \\ \mathscr{M}_\phi = \langle\{\phi(\vec{x})\},\infty|\Omega\rangle\ , \\ \left\langle\{\phi'(\vec{x})\},\infty\right|\mathscr{T}\left\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\right\}|\{\phi(\vec{x})\},\infty\rangle = \tilde{N}\int_{\phi(\vec{x},-\infty)=\phi(x)}^{\phi(\vec{x},+\infty)=\phi'(x)}\mathcal{D}\phi(\vec{x},t)e^{iS[\phi]}\prod_j\phi(x_j)\,, \end{gathered} \tag{16.22}$$

the expression can be written as

$$\int \mathcal{D}\phi(\vec{x},t)\mathscr{M}^*_{\phi'}\mathscr{M}_\phi e^{iS[\phi]}\prod_j \phi(x_j)\,. \tag{16.23}$$

The integral region is unconfined. In Weinberg's book [2], for the factor $\mathscr{M}^*_{\phi'}\mathscr{M}_\phi$, we can argue that the final effect is adding the $-i\epsilon\phi^2$ term to $\mathscr{L}$. So we conclude that

$$\langle\Omega|\,\mathscr{T}\left\{\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\right\}|\Omega\rangle = \frac{\int \mathcal{D}\phi(\vec{x},t)e^{iS[\phi]}\prod_j \phi(x_j)}{\int \mathcal{D}\phi(\vec{x},t)e^{iS[\phi]}}\,, \tag{16.24}$$

with normalization $\langle\Omega|\Omega\rangle = 1$ and $\mathscr{L}$ is modified by $-i\epsilon\phi^2$ term.

16.3 The generating functional (and how to compute)

Define generating functional (also partition function coupled to an external source,

$$Z[J] = \int \mathcal{D}\phi \exp\left[iS[\phi] + i\int d^4x J(x)\phi(x)\right]\,, \tag{16.25}$$

where $J(x)$ is the external current. We define the functional derivative as derivative satisfying

$$\frac{\delta}{\delta J(y)}J(x) = \delta^{(4)}(x-y)\,, \tag{16.26}$$

such that

$$\frac{\delta}{\delta J(y)}\int d^4x\phi(x)J(x) = \phi(y)\,. \tag{16.27}$$

Then

$$\Rightarrow \quad \langle\Omega|\,\mathscr{T}\left\{\prod_j \phi(x_j)\right\}|\Omega\rangle = (-i)^n\,\frac{1}{Z[0]}\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0}\,. \tag{16.28}$$

16.3.1 Computing $Z[J]$ in free theory - Gaussian integral

In free theory,

$$Z[J] = \int \mathcal{D}\phi \exp\left[i\int d^4x\left(-\frac{1}{2}\phi\left(\Box + m^2 - i\epsilon\right)\phi + J(x)\phi(x)\right)\right]\,. \tag{16.29}$$

Evaluate this expression by taking $n\to\infty$ limit, with formula

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{i=1}^n dx_i e^{-\frac{1}{2}x_iA_{ij}x_j + J_ix_i} = \sqrt{\frac{(2\pi)^n}{\det A}}e^{\frac{1}{2}J_i(A^{-1})_{ij}J_j}\,, \tag{16.30}$$

Remark 16.2. *Here we identify the ϕ and x label here.*

here $A = i(\Box + m^2 - i\epsilon)$ (required for convergence of Gaussian), $J_i \mapsto iJ(x)$. A^{-1} defined via

$$i(\Box_x + m^2 - i\epsilon)A^{-1} = \delta(x-y) \Rightarrow \quad A^{-1} = i\pi(x-y) = \int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m^2+i\epsilon}e^{ip(x-y)}\,, \tag{16.31}$$

which is the green function. The final result is

$$Z[J] = Z[0] \exp \left[-i \int d^4x \int d^4y \frac{1}{2} J(x) \pi(x-y) J(y) \right] . \tag{16.32}$$

16.3.2 Feynman rules for free theory via path integral

The two point function is

$$\begin{aligned} \langle 0 | \mathscr{T} \{ \phi(x_1) \phi(x_2) \} | 0 \rangle &= (-i)^2 \frac{1}{Z[0]} \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} Z[J] \bigg|_{J=0} \\ &= (-i)^2 \frac{\delta}{\delta J(x_2)} \frac{1}{Z[0]} (-i) \int d^4x J(x) \pi(x - x_1) Z[J] \bigg|_{J=0} \\ &= i\pi(x_1 - x_2) , \end{aligned} \tag{16.33}$$

which exactly reproduces the results from canonical propagator.

16.3.3 Evaluation of path integral in interacting theory

In the interacting theory,

$$\begin{aligned} \langle \Omega | \mathscr{T} \left\{ \hat{\phi}(x_1) \dots \hat{\phi}(x_n) \right\} | \Omega \rangle &= \frac{\int \mathcal{D}\phi(\vec{x}, t) e^{iS[\phi]} \prod_j \phi(x_j)}{\int \mathcal{D}\phi(\vec{x}, t) e^{iS[\phi]}} \\ &= \frac{\mathcal{D}\phi e^{i(S_0 + S_{\text{int}})} \prod_j \phi(x_j)}{\int \mathcal{D} e^{i(S_0 + S_{\text{int}})}} \\ &= \frac{\int \mathcal{D}\phi e^{iS_0} \prod \phi(x_i) e^{iS_{\text{int}}}}{\int \mathcal{D}\phi e^{iS_0}} \frac{\int \mathcal{D}\phi e^{iS_0}}{\int \mathcal{D}\phi e^{iS_0} e^{iS_{\text{int}}}} \\ &= \frac{\langle 0 | \mathscr{T} \left\{ \prod_i \hat{\phi}(x_i) e^{iS_{\text{int}}} \right\} | 0 \rangle}{\langle 0 | \mathscr{T} \{ e^{iS_{\text{int}}} \} | 0 \rangle} . \end{aligned} \tag{16.34}$$

16.4 Fermionic path integral

The essential ingredient in derivation of bosonic path integral is

- $\exists$ a set of eigenbasis of $\mathscr{H}$ of $\hat{x}, \hat{p}$, with $[\hat{x}, \hat{p}] = i$, have the complete relation

$$\mathbb{1} = \int dx \, |x\rangle \langle x| = \int dp \, |p\rangle \langle p| . \tag{16.35}$$

- $\langle p | x \rangle \propto e^{-ipx}$.

In fermionic theory, the canonical operators $\hat{q}_i$, $\hat{p}_j$ satisfy anticommutation relations $\{\hat{q}_i, \hat{p}_j\} = i\delta_{ij}$, $\{\hat{q}_i, \hat{q}_j\} = \{\hat{p}_i, \hat{p}_j\} = 0$.

Remark 16.3. *No non-trivial eigenstate to $\hat{q}_i$, $\hat{p}_i$.*

Proof: Assume that $|q_1, \dots, q_n\rangle$ is such a state, then

$$\hat{q}_i \hat{q}_j |q_1, \dots, q_n\rangle = \hat{q}_i q_j |q_1, \dots, q_n\rangle = q_j q_i |q_1, \dots, q_n\rangle , \tag{16.36}$$

in another direction

$$\hat{q}_i\hat{q}_j |q_1,\ldots,q_n\rangle = -\hat{q}_j\hat{q}_i |q_1,\ldots,q_n\rangle = -q_iq_j |q_1,\ldots,q_n\rangle \tag{16.37}$$

$$\Rightarrow \quad q_iq_j = 0\,, \tag{16.38}$$

where there indeed exist a problem.

16.4.1 Construction of representation space of fermionic operator algebra

- Let $|0\rangle$ be annihilated by all $\hat{q}_i$, $\hat{q}_i|0\rangle = 0$.
- Define $\mathscr{H}_f = \langle |0\rangle, \hat{p}_i|0\rangle, \hat{p}_i\hat{p}_j|0\rangle, \ldots, \hat{p}_1\ldots\hat{p}_n|0\rangle \,|i<\ldots<j\rangle_{\mathbb{C}}$. $\mathscr{H}_f$ is clearly closed under the action of $\hat{q}_i$ and $\hat{p}_i$. To organize states fo $\mathscr{H}_f$ in terms of eigenbasis to $\hat{q}_i$, $\hat{p}_i$. We will introduce anti-commutating variables.

16.4.2 Grassmann algebra

A Grassmann algebra G_q over $\mathbb{C}$:

- a complex vector space generated freely by elements $q_i, q_iq_j, q_1\ldots q_n$: $1\le i<\ldots<j\le n$.
- an associate anti-commutative product induced by $q_i\times q_j = -q_j\times q_i$ with $q_i\times\ldots\times q_j = q_i\ldots q_j$ with $i<\ldots<j$.

We then introduce $\mathscr{H}_f^q = G_q\otimes_{\mathbb{C}}\mathscr{H}_f$. Let $\{\hat{q}_1,\ldots,\hat{q}_n,\hat{p}_1,\ldots,\hat{p}_n\}$ act on $\mathscr{H}_f^q$ by imposing $\{q_i,\hat{q}_j\} = \{q_i,\hat{p}_j\} = 0$, such that

$$\hat{p}_i\hat{p}_j(q_k|0\rangle) = q_k(\hat{p}_i\hat{p}_j|0\rangle) \in \mathscr{H}_f^q. \tag{16.39}$$

16.4.3 Eigenvectors to $\hat{q}_i$ in $\mathscr{H}_f^q$

We define

$$\begin{aligned} |q\rangle &:= \exp\left[-i\sum_{j=1}^n \hat{p}_jq_j\right]|0\rangle \\ &= \left[1 - i\sum_{j=1}^n\hat{p}_jq_j + \ldots + \frac{(-i)^n}{n!}\left(\sum_{j=1}^n\hat{p}_jq_j\right)^n\right]|0\rangle\,. \end{aligned} \tag{16.40}$$

The higher order expansion vanishes because of the Grassmann variables. This is a single state, i.e. q in $|q\rangle$ does not vary.

$$\begin{aligned} \hat{q}_i|q\rangle &= \hat{q}_i\exp\left[-i\sum_{j=1}^n\hat{p}_jq_j\right]|0\rangle = \hat{q}_i\exp(-i\hat{p}_iq_i)\exp\left(-i\sum_{j\neq i}\hat{p}_jq_j\right)|0\rangle \\ &= \hat{q}_i(1-i\hat{p}_iq_i)\exp\left(-i\sum_{j\neq i}\hat{p}_jq_j\right)|0\rangle = (\hat{q}_i - i\hat{q}_i\hat{p}_iq_i)\exp\left(-i\sum_{j\neq i}\hat{p}_jq_j\right)|0\rangle \\ &= (\hat{q}_i - i\{\hat{q}_i,\hat{p}_i\}q_i + i\hat{p}_i\hat{q}_iq_i)\exp\left(-i\sum_{j\neq i}\hat{p}_jq_j\right)|0\rangle\,, \end{aligned} \tag{16.41}$$

since $\hat{q}_i |0\rangle = 0$, and $\hat{q}_i$ anticommutes with the rest variable, we left with

$$\hat{q}_i |q\rangle = q_i \exp\left(-i\sum_{j\neq i}\hat{p}_j q_j\right) . \tag{16.42}$$

Because $q_i^2 = 0$, we can insert the term

$$\begin{aligned}\hat{q}_i |q\rangle &= q_i(1 - i\hat{p}_i q_i)\exp\left(-i\sum_{j\neq i}\hat{p}_j q_j\right)|0\rangle \\ &= q_i \exp\left(-i\sum_{j=1}^{n}\hat{p}_j q_j\right)|0\rangle = q_i |q\rangle .\end{aligned} \tag{16.43}$$

To define the eigenstates for the $\hat{p}_i$, we introduce a second set of Grassmann generators $\{p_1, \ldots, p_n\}$ with $\{q_i, p_j\} = \{\hat{q}_i, p_j\} = \{\hat{p}_i, p_j\} = 0$. We now work in the Hilbert space

$$\mathscr{H}_f^{q,p} = G_{q,p} \otimes_{\mathbb{C}} \mathscr{H}_f . \tag{16.44}$$

The eigenstates for $\hat{p}_i$ defined by

$$|p\rangle := \exp\left(-i\sum_{i=1}^{n}\hat{q}_i p_i\right)\prod_{i=1}^{N}\hat{p}_i |0\rangle , \tag{16.45}$$

with $\hat{p}_j \prod_{i=1}^{n}\hat{p}_i |0\rangle = 0$. The same discussion will produce

$$\hat{p}_i |0\rangle = p_i |p\rangle . \tag{16.46}$$

Finally, define the dual Hilbert space built on $\langle 0|$, $\langle 0|\hat{p}_i = 0$ for $\forall i$, and $\langle 0|0\rangle = 1$ with basis

$$\{\langle 0| , \langle 0|\hat{q}_i, \langle 0|\hat{q}_i\hat{q}_j, \ldots, \langle 0|\hat{q}_1 \ldots \hat{q}_n | 1 \le i < \ldots < j \le n\} . \tag{16.47}$$

And eigenstate

$$\begin{aligned}\langle q| &= \langle 0|\prod_{i=1}^{n}\hat{q}_i \exp\left(-i\sum q_i\hat{p}_i\right) , \quad \langle q|\hat{q}_i = \langle q| q_i , \\ \langle p| &= \langle 0|\exp\left(-i\sum p_i\hat{q}_i\right) , \quad \langle p|\hat{p}_i = \langle p| p_i .\end{aligned} \tag{16.48}$$

We need to evaluate $\langle q|p\rangle$,

$$\begin{aligned}\langle q|p\rangle &= \langle q|\exp\left(-i\sum_{i=1}^{n}\hat{q}_i p_i\right)\prod_{i=1}^{n}\hat{p}_i|0\rangle = \exp\left(-i\sum_{i=1}^{n}q_i p_i\right)\langle q|\prod_{i=1}^{n}\hat{p}_i|0\rangle \\ &= \exp\left(-i\sum_{i=1}^{n}q_i p_i\right)\langle 0|\prod_{i=1}^{n}\hat{q}_i \exp\left(-i\sum q_i\hat{p}_i\right)\prod_{i=1}^{n}\hat{p}_i|0\rangle ,\end{aligned} \tag{16.49}$$

because of the fermionic property, only the zero order term in the exponential left, then

$$\langle q|p\rangle = \exp\left(-i\sum_{i=1}^{n} q_i p_i\right)\langle 0|\prod_{i=1}^{n}\hat{q}_i\prod_{i=1}^{n}\hat{p}_i\,|0\rangle = \chi_n \exp\left(-i\sum_{i=1}^{n} q_i p_i\right)\,, \tag{16.50}$$

where χ_n is the phase factor depends on n. Likewise,

$$\langle p|q\rangle \propto e^{-i\sum p_i q_i}\,. \tag{16.51}$$

16.4.4 Berezin integration

Note that both $|q\rangle$ and $|p\rangle$ are linear combinations (with Grassmann coefficients) of all the states

$$|0\rangle\,,\, \hat{p}_i\,|0\rangle\,,\,\ldots,\, \hat{p}_1\ldots\hat{p}_n\,|0\rangle\ . \tag{16.52}$$

to access a given state in $\mathscr{H}_f$, we want a tool which isolates a given coefficient of Grassmann generators.

Remark 16.4. *In usual Hilbert space, innerproduct with the orthogonal basis can give us the coefficients.*

Definition 16.1. $\int dq_{j_1}\ldots dq_{j_k}$ *acts on* $\mathscr{H}_f^{q,p}\otimes\mathscr{H}_{boson}$ *via*

$$\int dq_1 f(q_1,\ldots,q_n) = \int dq_1\left(c_0(q_2,\ldots,q_n) + q_1 c_1(q_2.\ldots,q_n)\right) = c_1(q_2,\ldots,q_n)\,, \tag{16.53}$$

and

$$\int dq_1\ldots\int dq_n f = \int dq_1\left(\ldots\left(\int dq_n f\right)\ldots\right)\,. \tag{16.54}$$

In particular,

$$\int dq_{j_1}\ldots dq_{j_m}\prod_{i\in I} q_i = 0,\ \text{if}\quad \{j_1,\ldots,j_m\}\not\subset I\,. \tag{16.55}$$

Example 16.1.

$$\int dq_1 dq_2\left(aq_1 + bq_1q_2 + cq_1q_2q_3\right) = -b - cq_3\,. \tag{16.56}$$

Theorem 16.1.

$$\mathbb{1} = \tilde{\chi}^n\int\prod_{i=1}^{n} dq_i\,|q\rangle\langle q|\ , \tag{16.57}$$

where $\tilde{\chi}^n$ *is the* n*-dependent phase. This is the fermionic completeness relation.*

Proof: Consider $|f\rangle = \hat{p}_1\ldots\hat{p}_k\,|0\rangle$ (without loss of generality),

$$\begin{aligned}\langle q|f\rangle &= \langle 0|\prod_{i=1}^{n}\hat{q}_i\exp\left(-i\sum q_i\hat{p}_i\right)\hat{p}_1\ldots\hat{p}_k\,|0\rangle\\ &= \langle 0|\prod_{i=1}^{n}\hat{q}_i\frac{(-i)^{n-k}}{(n-k)!}\left(\sum_{i=1}^{n} q_i\hat{p}_i\right)^{n-k}\hat{p}_1\ldots\hat{p}_k\,|0\rangle\ ,\end{aligned} \tag{16.58}$$

this step needs a little more explanation. First, since we have $\hat{p}_i$ for $i\le k$, if the order is lager that $n-k$, there must exists m $(m\le k)$ such that $\hat{p}_m$ appears twice at least, causing the expression vanished. If the order is less than $n-k$, there must exists l such that $\hat{p}_l$ doesn't appear in the expression. However, $\hat{q}_l$ anticommutes with all the other terms, and we could move it to the position next to the $|0\rangle$ (up to a sign difference), annihilating the state. So only the $(n-k)$-th order term survives. Continue the calculation

$$= (-i)^{n-k} \langle 0| \prod_{i=1}^{n} \hat{q}_i \hat{p}_1 \ldots \hat{p}_k q_{k+1} \hat{p}_{k+1} \ldots q_n \hat{p}_n |0\rangle \ , \tag{16.59}$$

where we use the property that $q_i \hat{p}_i$ moves as a block, which doesn't change the sign. So

$$= (-i)^{n-k} (-1)^{1+2+\ldots+(n-k)} \langle 0| \prod_{i=1}^{n} \hat{q}_i \hat{p}_1 \ldots \hat{p}_n |0\rangle \, q_{k+1} \ldots q_n \ . \tag{16.60}$$

We set $A = 1+2+\ldots+(n-k)$ and $\chi_n = \langle 0| \prod_{i=1}^{n} \hat{q}_i \hat{p}_1 \ldots \hat{p}_n |0\rangle$, substitute into the integral, we have

$$(-i)^{n-k} (-1)^{A} \chi_n \int \prod_{i=1}^{n} dq_i \, |q\rangle \, q_{k+1} \ldots q_n \ , \tag{16.61}$$

with definition of $|q\rangle = \exp\left(-i \sum_j \hat{p}_j q_j\right) |0\rangle$, only the k-th order terms survive. We have

$$\begin{aligned} &(-i)^{n-k} (-1)^{A} \chi_n \int \prod_{i=1}^{n} dq_i (-i)^k \hat{p}_1 q_1 \ldots \hat{p}_k q_k |0\rangle \, q_{k+1} \ldots q_n \\ =\ &(-i)^{n} (-1)^{A} \chi_n \int \prod_{i=1}^{n} dq_i (-1)^{(n-k+1)+\ldots+n} q_1 \ldots q_n \hat{p}_1 \ldots \hat{p}_k |0\rangle \\ =\ &(-i)^{n} (-1)^{\sum_{j=1}^{n} j} \chi_n \int \prod_{i=1}^{n} dq_i (q_1 \ldots q_n) \, |f\rangle = \tilde{\chi}_n |f\rangle \ . \end{aligned} \tag{16.62}$$

We finish the proof.

16.4.5 Fermionic matrix elements via path integral

Introduce $\hat{q}_i(t) = e^{iHt} \hat{q}_i e^{-iHt}$, $\hat{p}_i(t) = e^{iHt} \hat{p}_i e^{-iHt}$, the eigenstate for these operators are

$$|q, t\rangle = e^{i\hat{H}t} |q\rangle \ , \ |p, t\rangle = e^{i\hat{H}t} |p\rangle \ . \tag{16.63}$$

Now we want to evaluate $\langle q', t_f | q, t_i \rangle$, where q' represents the second set of Grassmann generators q'_i. The strategy is similar to the bosonic case. Divide the finite time into infinitesimal piece $t_f - t_i = (N+1)\Delta t$, introduce the completeness relation for a set of Grassmann variables $\{q_i^k\}_{i=1,\ldots,n}$ at each point $t_k = t_i + k\Delta t$. Then

$$\langle q', t_f | q, t_i \rangle = \int \prod_{i,k} dq_i^k dp_i^k \prod_{i=0}^{N} \exp\left[i\Delta t \sum_i \left(p_i^l \frac{q_i^{l+1} - q_i^l}{\Delta t} - H(p^l, q^l) \right) \right] \ . \tag{16.64}$$

Now take the limit $\Delta t \to 0$, and introduce the independent Grassmann generators indexed by a continuous variable t:

$$\{q_i^k\}_{i=1,\ldots,n} \to \{q_i(t)\}_{i=1,\ldots,n} \quad \{p_i^k\} \to \{p_i(t)\} \ , \tag{16.65}$$

such that

$$\langle q', t_f | q, t_i \rangle = \int \prod_{i=1}^{n} \mathcal{D}q_i(t) \prod_{i=1}^{n} \mathcal{D}p_i(t) \exp\left[i \int_{t_i}^{t_f} dt \left(\sum_i p_i(t)\dot{q}_i(t) - H(p_i(t), q_i(t)) \right) \right] . \tag{16.66}$$

Remark 16.5. *Unlike the bosonic case, in which H is quadratic in the canonical variable p. Because the property of the Grassmann number, we do not perform the $\mathcal{D}p_i(t)$ integral as H, L are generically linear in p, q.*

16.4.6 Evaluation of fermionic path integral

Define generating function by coupling to fermionic (i.e. anti-commuting) currents,

$$Z[j^p, j^q] = \int \prod \mathcal{D}q_i(t) \prod \mathcal{D}p_i(t) \exp\left(iS + i \int dt \sum_i (p_i j_i^p + j_i^q q_i) \right) . \tag{16.67}$$

Besides, we introduce derivation with Grassmann variables,

$$\begin{aligned} \frac{\partial}{\partial q_i} q_1 \dots q_n &= (-i)^{i-1} q_1 \dots q_{i-1} \frac{\partial}{\partial q_i} q_i q_{i+1} \dots q_n \\ &= (-1)^{i-1} q_1 \dots q_{i-1} q_{i+1} \dots q_n . \end{aligned} \tag{16.68}$$

The variation with respect to the current is

$$\begin{aligned} &\frac{i\delta}{\delta j_1^p(t_1)} \cdots \frac{i\delta}{\delta j_m^p(t_m)} \frac{-i\delta}{\delta j_{m+1}^q(t_{m+1})} \cdots \frac{-i\delta}{\delta j_k^q(t_k)} Z[j^p, j^q] = \\ &\int \prod \mathcal{D}q_i(t) \prod \mathcal{D}p_i(t) \prod_{r=1}^{m} p_r(t_r) \prod_{s=m+1}^{k} q_s(t_s) \exp\left[iS + i \int dt \sum_i (p_i j_i^p + j_i^q q_i) \right] . \end{aligned} \tag{16.69}$$

$Z[j^p, j^q]$ can be evaluated for S of the form $-p_i A_{ik} q_k$,

$$\begin{aligned} &\int \prod dq_i \prod dp_i \exp\left(-p_i A_{ik} q_k + p_i j_i^p + j_i^q q_i\right) \\ &= \int \prod dq_i \prod dp_i \exp\left(-(p - j^q A^{-1})_i A_{ik} (q - A^{-1} j^p)_k + j_i^q A_{ik}^{-1} j_k^p \right) . \end{aligned} \tag{16.70}$$

Notice that for $f(q) = a + bq$, we have

$$\int dq f(q + j) = \int dq (a + b(q + j)) = \int dq f(q) , \tag{16.71}$$

we have

$$\int \prod_{i=1}^{n} dq_i \prod_{i=1}^{n} dp_i e^{-p_i A_{ik} q_k} = \frac{(-1)^n}{n!} \int \prod dq_i \prod dp_i (p_i A_{ik} q_k)^n , \tag{16.72}$$

where only the n order left because of the integration property. Here the repeated i, k indices mean the summation. Rearranging the summation order, because $p_i A_{ik} q_k$ moves as a block and doesn't change the sign, we can rewrite the expression.

$$
\begin{aligned}
&= \frac{(-1)^n}{n!}\int \prod dq_i \prod dp_i n! \sum_{\{i_1,\dots,i_n\}} (p_n A_{n,i_n} q_{i_n}) \dots (p_1 A_{1,i_1} q_{i_1}) \\
&= (-1)^n (-1)^{\frac{(n-1)n}{2}} \int \prod dq_i \prod dp_i p_n p_{n-1} \dots p_1 \sum_{\{i_1,\dots,i_n\}} A_{n,i_n} \dots A_{1,i_1} q_{i_n} \dots q_{i_1} \\
&= (-1)^n (-1)^{\frac{(n-1)n}{2}} \int \prod dq_i \sum_{\{i_1,\dots,i_n\}} A_{n,i_n} \dots A_{1,i_1} q_{i_n} \dots q_{i_1} \\
&= (-1)^n (-1)^{\frac{(n-1)n}{2}} \int \prod dq_i \sum_{\{i_1,\dots,i_n\}} A_{n,i_n} \dots A_{1,i_1} (-1)^\sigma q_n \dots q_1 \\
&= (-1)^{\frac{n(n+1)}{2}} \det A \,,
\end{aligned}
\tag{16.73}
$$

where σ is the number of the transposition, which can combine with the element of A and give the determinant of A. So the integral is

$$
\int \prod dq_i \prod dp_i e^{-p_i A_{ik} q_k + p_i j_i^p + j_i^q q_i} = (-1)^{\frac{n(n+1)}{2}} \det A e^{j_i^q (A^{-1})_{ik} j_k^p} \,. \tag{16.74}
$$

16.4.7 The fermionic path integral in QFT

At a given time interval, we introduce a continuous family of Grassmann variables $\{\psi(\vec{x},t)\}$, indexed by the spatial vector $\vec{x} \in \mathbb{R}^3$, with conjugate variables $\{\psi^\dagger(\vec{x},t)\}$. The generating functional is

$$
\begin{aligned}
Z[\bar{\eta},\eta] &= \int \mathcal{D}\bar{\psi}(\vec{x},t)\mathcal{D}\psi(\vec{x},t) \exp\left[i \int d^4x \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta\right] \\
&= Z[0] \exp\left[-i \int d^4x \int d^4y \bar{\eta}(y)(i\partial\!\!\!/ - m)^{-1}\eta(x)\right] \,,
\end{aligned}
\tag{16.75}
$$

where $Z[0] = N \det\big(i\partial\!\!\!/ - m\big)$ and $(i\partial\!\!\!/ - m)^{-1}$ is the Dirac propagator.

16.5 Applications

16.5.1 The photon propagator

To obtain the propagator, we need to insert the photon kinetic term, which is

$$
\begin{aligned}
-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \xrightarrow{\text{momentum space}} &\frac{1}{4}(k_\mu A_\nu - k_\nu A_\mu)(k^\mu A^\nu - k^\nu A^\mu) \\
&= \frac{1}{2}(k^2 A^2 - (k \cdot A)^2) \\
&= -\frac{1}{2} A^\mu \left(-k^2 g_{\mu\nu} + k_\mu k_\nu\right) A^\nu \,,
\end{aligned}
\tag{16.76}
$$

however, the complexity raises here because $-k^2 g_{\mu\nu} + k_\mu k_\nu$ is not invertible. It is easy to justify because the eigenvector k^μ has eigenvalue 0. This is the consequence of the gauge invariance. A perturbation of this operator renders it invertible.

$$-k^2 g_{\mu\nu} + k_\mu k_\nu \longrightarrow -k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu \,. \tag{16.77}$$

With inverse

$$\Pi_{\mu\nu} = -\frac{g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2} \,, \tag{16.78}$$

which is exactly the photon propagator in ξ-gauge. We are going the explain where the perturbation comes from via path integral. How to justify the perturbation? It can be introduced by adding $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ to the Lagrangian.

Theorem 16.2. *This term doesn't modify the time order product of gauge invariant operator.*

Proof: Consider the function

$$f(\xi) = \int \mathcal{D}\pi e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi)^2} \,. \tag{16.79}$$

For fixed $A^\mu(x)$, we perform the change of variables

$$\pi(x) \to \pi(x) - \frac{1}{\Box}\partial_\mu A^\mu \,, \tag{16.80}$$

where the last term represents the solution for $\Box\alpha = \partial_\mu A^\mu$. This is a shift. The measure will run through all the possible fields so the measure won't change. Then

$$f(\xi) = \int \mathcal{D}\pi e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2} \,. \tag{16.81}$$

It looks like that it depends on A^μ, but the integral doesn't because A^μ doesn't appear in the initial definition of $f(\xi)$. Next, let $\hat{\mathscr{O}}(x_1, \ldots, x_n)$ be a gauge invariant product of operator. We consider

$$\langle\Omega| \mathscr{T}\left\{\hat{\mathscr{O}}(x_1,\ldots,x_n)\right\}|\Omega\rangle = \frac{1}{Z[0]}\int \mathcal{D}A \prod \mathcal{D}\phi_i \prod \mathcal{D}\phi_i^* e^{i\int d^4x \mathscr{L}[A,\phi_i,\phi_i^*]} \mathscr{O}(x_1,\ldots,x_n) \,, \tag{16.82}$$

where ϕ_i and ϕ_i^* are both charged fields. Insert the function $f(\xi)$, we have

$$= \frac{1}{Z[0]f(\xi)}\int \mathcal{D}\pi\mathcal{D}A \prod \mathcal{D}\phi_i \prod \mathcal{D}\phi_i^* e^{i\int d^4x \mathscr{L}[A,\phi_i,\phi_i^*] - \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2} \mathscr{O}(x_1,\ldots,x_n) \,. \tag{16.83}$$

To decouple π and A^μ, we perform a gauge transformation with gauge parameter π,

$$A_\mu \to A_\mu + \partial_\mu\pi, \quad \phi_i = e^{iQ_i\pi}\phi_i \,. \tag{16.84}$$

By gauge invariance of $\mathscr{L}$ and $\mathscr{O}$, the integral becomes,

$$\begin{gathered}\langle\Omega| \mathscr{T}\left\{\hat{\mathscr{O}}(x_1,\ldots,x_n)\right\}|\Omega\rangle = \\ \frac{1}{Z[0]f(\xi)}\int \mathcal{D}\pi\mathcal{D}A \prod \mathcal{D}\phi_i \prod \mathcal{D}\phi_i^* e^{i\int d^4x \mathscr{L}[A,\phi_i,\phi_i^*] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2} \mathscr{O}(x_1,\ldots,x_n) \,.\end{gathered} \tag{16.85}$$

Performing the same manipulation for $Z[0]$ yields,

$$Z[0] = \frac{1}{f(\xi)} \int \mathcal{D}\pi \mathcal{D}A \prod \mathcal{D}\phi_i \prod \mathcal{D}\phi_i^* e^{i\int d^4x \mathscr{L}[A,\phi_i,\phi_i^*] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2} . \tag{16.86}$$

The factor $\frac{1}{f(\xi)}\mathcal{D}\pi$ cancels and the Lagrangian is modified

$$\mathscr{L}[A,\phi_i,\phi_i^*] \to \mathscr{L}[A,\phi_i,\phi_i^*] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 . \tag{16.87}$$

16.5.2 The Ward-Takahashi identity

Consider the theory with global symmetry $\psi \to e^{i\alpha}\psi$ for ψ, which is a Dirac spinor. The propagator

$$\langle\Omega| \mathscr{T}\left\{\psi(x_1)\bar\psi(x_2)\right\}|\Omega\rangle = \frac{1}{Z[0]}\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}X \exp\left[i\int d^4x\left(\bar\psi(i\partial\!\!\!/ - m)\psi + Y\right)\right]\psi(x_1)\bar\psi(x_2) , \tag{16.88}$$

where $\mathcal{D}X$ is the measure part of the other fields which aren't relate to the Dirac spinor. Y is the Lagrangian term doesn't include $\psi, \bar\psi$. Consider field redefinition,

$$\begin{gathered}
\psi(x) \to e^{-i\alpha(x)}\psi \\
\mathcal{D}\psi\mathcal{D}\bar\psi \to \mathcal{D}\psi\mathcal{D}\bar\psi \\
\bar\psi(i\partial\!\!\!/ - m)\psi \to \bar\psi(i\partial\!\!\!/ - m)\psi + \bar\psi\gamma^\mu\psi\partial_\mu\alpha(x) \\
\mathcal{D}X, Y \to \mathcal{D}X, Y \\
\psi(x_1)\bar\psi(x_2) \to e^{-i\alpha(x_1)+i\alpha(x_2)}\psi(x_1)\bar\psi(x_2) .
\end{gathered} \tag{16.89}$$

Now expand in α. Because the theory has global symmetry, now the $\alpha(x)$ transformation is local, and the higher derivative (higher than zero order) part should be zero.

$$\frac{1}{Z[0]}\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}X e^{iS} \exp\left[i\int d^4x\bar\psi\gamma^\mu\psi\partial_\mu\alpha(x) - i\alpha(x_1) + i\alpha(x_2)\right]\psi(x_1)\bar\psi(x_2) = 0 . \tag{16.90}$$

With definition $j^\mu = \bar\psi\gamma^\mu\psi$, the part in the exponential function can be rewritten as

$$\left[i\int d^4x\bar\psi\gamma^\mu\psi\partial_\mu\alpha(x) - i\alpha(x_1) + i\alpha(x_2)\right] = -i\int d^4x\alpha(x)\left[\partial_\mu j^\mu(x) + \delta^{(4)}(x-x_1) - \delta^{(4)}(x-x_2)\right] , \tag{16.91}$$

$\alpha(x)$ is arbitrary, then we have the following relation

$$\frac{1}{Z[0]}\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}X e^{iS}\partial_\mu j^\mu\psi(x_1)\bar\psi(x_2) = \frac{1}{Z[0]}\left[-\delta(x-x_1) + \delta(x-x_2)\right]\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}X e^{iS}\psi(x_1)\bar\psi(x_2) , \tag{16.92}$$

which is equivalent to

$$\Leftrightarrow \partial_\mu\langle\Omega|\mathscr{T}\left\{j^\mu\psi(x_1)\bar\psi(x_2)\right\}|\Omega\rangle = \left[-\delta(x-x_1) + \delta(x-x_2)\right]\langle\Omega|\mathscr{T}\left\{\psi(x_1)\bar\psi(x_2)\right\}|\Omega\rangle . \tag{16.93}$$

Introducing the Fourier transformation,

$$\begin{aligned}\mathscr{M}^\mu(p,q_1,q_2) &= \int d^4x \int d^4x_1 \int d^4x_2 e^{ipx+iq_1x_1+iq_2x_2} \langle\Omega|\,\mathscr{T}\left\{j^\mu\psi(x_1)\bar{\psi}(x_2)\right\}|\Omega\rangle \\ \mathscr{M}(q_1,q_2) &= \int d^4x_1 \int d^4x_2 e^{iq_1x_1+iq_2x_2} \langle\Omega|\,\mathscr{T}\left\{\psi(x_1)\bar{\psi}(x_2)\right\}|\Omega\rangle\,.\end{aligned} \tag{16.94}$$

After Fourier transformation,

$$\begin{aligned}\Rightarrow \quad & \int d^4x \int d^4x_1 \int d^4x_2 e^{ipx+iq_1x_1+iq_2x_2}\partial_\mu \langle\Omega|\,\mathscr{T}\left\{j^\mu\psi(x_1)\bar{\psi}(x_2)\right\}|\Omega\rangle = -ip_\mu\mathscr{M}^\mu(p,q_1,q_2) \\ &= \int d^4x \int d^4x_1 \int d^4x_2 e^{ipx+iq_1x_1+iq_2x_2}[-\delta(x-x_1)+\delta(x-x_2)] \langle\Omega|\,\mathscr{T}\left\{\psi(x_1)\bar{\psi}(x_2)\right\}|\Omega\rangle \\ &= \int d^4x_1 d^4x_2 \left[-e^{i(p+q_1)x_1+iq_2x_2} + e^{iq_1x_1+i(p+q_2)x_2}\right] \langle\Omega|\,\mathscr{T}\left\{\psi(x_1)\bar{\psi}(x_2)\right\}|\Omega\rangle \\ &= -\mathscr{M}(p+q_1,q_2) + \mathscr{M}(q_1,p+q_2)\,.\end{aligned} \tag{16.95}$$

We get the Ward-Takahashi identity

$$ip_\mu\mathscr{M}^\mu(p,q_1,q_2) = \mathscr{M}(q_1+p,q_2) - \mathscr{M}(q_1,q_2+p)\,. \tag{16.96}$$

Resuming the argument with the insertion

$$\mathscr{O}^{\mu_1\ldots\mu_n} = \prod_i j^{\mu_i}(x_i) \prod_i \psi(y_i) \prod_i \bar{\psi}(z_i)\,. \tag{16.97}$$

The corresponding Fourier transformation variables are $x_i \leftrightarrow p_i$, $y_i \leftrightarrow q_i$, $z_i \leftrightarrow r_i$, and define

$$\begin{aligned}&\mathscr{M}^{\mu\mu_1\ldots\mu_k}(p,p_1,\ldots,p_k,q_1,\ldots,q_l,r_1,\ldots,r_m) = \\ &\int d^4x d\mu e^{ipx+i\sum_i p_ix_i+i\sum_i q_iy_i+i\sum r_iz_i} \langle\Omega|\,\mathscr{T}\left\{j^\mu(x)\mathscr{O}^{\mu_1\ldots\mu_n}\right\}|\Omega\rangle\,,\end{aligned} \tag{16.98}$$

where $d\mu = \prod d^4x_i \prod d^4y_i \prod d^4z_i$. And

$$\mathscr{M}^{\mu_1\ldots\mu_k}(p_1,\ldots,p_k,q_1,\ldots,q_l,r_1,\ldots,r_m) = \int d\mu e^{i\sum_i p_ix_i+i\sum_i q_iy_i+i\sum r_iz_i} \langle\Omega|\,\mathscr{T}\left\{\mathscr{O}^{\mu_1\ldots\mu_n}\right\}|\Omega\rangle\,. \tag{16.99}$$

Those yields the generalized Ward-Takahashi identity

$$\begin{aligned}ip_\mu\mathscr{M}^\mu\mu_1\ldots\mu_k &= ip_\mu\mathscr{M}^{\mu\mu_1\ldots\mu_k}(p,p_1,\ldots,p_k,q_1,\ldots,q_l,r_1,\ldots,r_m) \\ &= \sum_{i=1}^{l}\mathscr{M}^{\mu_1\ldots\mu_k}(p_1,\ldots,p_k,q_1,\ldots,q_i+p,\ldots,q_l,r_1,\ldots,r_m) - \\ &\sum_{i=1}^{m}\mathscr{M}^{\mu_1\ldots\mu_k}(p,p_1,\ldots,p_k,q_1,\ldots,q_l,r_1,\ldots,r_i+p,\ldots,r_m)\,.\end{aligned} \tag{16.100}$$

Note that the momentum p_i associated to the current insertions are not shifted, as these are invariant under the action $\psi \to e^{-i\alpha(x)}\psi$.

16.5.3 Reduction to Ward identity in QED

In the process of reducing the Ward-Takahashi identity to Ward identity, replacing the photon polarization tensor $\varepsilon^{\mu}(p)$ in our S-matrix element by p^{μ} yields zero. Recall that $\varepsilon^{\mu}(p)$ and $\varepsilon^{\mu}(p)+cp^{\mu}$ should be physically equivalent required by the gauge invariance. We now prove the Ward identity using Ward-Takahashi identity now.

Proof:
we will focus on outgoing photons for simplifying the notation. Consider an S-matrix with $k+1$ such photons, and an arbitrary number (g) of in and outgoing electrons and positrons. By LSZ reduction formula, we have

$$\begin{aligned} S = \varepsilon_{\mu}(p)\prod_{i=1}^{k}\varepsilon_{\mu_i}(p_i)&\left[i\int d^4x e^{ipx}\Box_x\left(\prod i\int d^4x_i e^{ip_ix_i}\Box_{x_i}\right)\times\dots\right] \\ &\times\langle\Omega|\,\mathscr{T}\left\{A^{\mu}(x)A^{\mu_1}(x_1)\dots A^{\mu_k}(x)X\right\}|\Omega\rangle\ , \end{aligned} \tag{16.101}$$

where X includes all the $\psi,\bar{\psi}$-dependent terms. In Lorenz gauge, the EoM of $A^{\mu}(x)$ are $\Box A^{\mu}=e\bar{\psi}\gamma^{\mu}\psi=j^{\mu}$. By Schwinger-Dyson equation

$$\begin{aligned} &\prod_{i=1}^{k}\Box_{x_k}\Box_x\langle\Omega|\,\mathscr{T}\left\{A^{\mu}(x)A^{\mu_1}(x_1)\dots A^{\mu_k}(x)X\right\}|\Omega\rangle \\ &=\prod_{i=1}^{k}\Box_{x_k}\langle\Omega|\,\mathscr{T}\left\{j^{\mu}(x)\prod_i A^{\mu_i}(x_i)X\right\}|\Omega\rangle \\ -i\sum_{j=1}&\delta^{(4)}(x-x_j)g^{\mu\mu_j}\langle\Omega|\,\mathscr{T}\left\{A^{\mu_1}(x_1)\dots\cancel{A^{\mu_j}(x_j)}\dots A^{\mu_k}(x_k)X\right\}|\Omega\rangle \\ &=\langle\Omega|\,\mathscr{T}\left\{j^{\mu}(x)\prod_i j^{\mu_i}(x_i)X\right\}|\Omega\rangle+\text{contact terms}\,. \end{aligned} \tag{16.102}$$

The connected S-matrix is

$$\begin{aligned} S_{\text{connected}}&=\varepsilon_{\mu}(p)\prod_{i=1}^{k}\varepsilon_{\mu_i}(p_i)i\int d^4x e^{ipx}\prod_{i=1}^{k}\int d^4x_i e^{ip_ix_i}\dots\langle\Omega|\,\mathscr{T}\left\{j^{\mu}(x)\prod_{i=1}^{k}j^{\mu_i}(x_i)X\right\}|\Omega\rangle \\ &=i^g\varepsilon_{\mu}(p)\prod_{i=1}^{k}\varepsilon_{\mu_i}(p_i)\prod_{i=1}^{l}(q_i^2-m^2)\prod_{i=1}^{\bar{l}}(r_i^2-m^2)\mathscr{M}^{\mu_1\dots\mu_k}(p,p_1,\dots,p_k,q_1,\dots,q_l,r_1,\dots,r_{\bar{l}})\,. \end{aligned} \tag{16.103}$$

The red part are the poles in the time order product. By replacing $\varepsilon\to ip_{\mu}$, we have

$$\begin{aligned} \prod_{i=1}^{k}\varepsilon_{\mu_i}(p_i)\prod_{i=1}^{l}(q_i^2-m^2)&\prod_{i=1}^{\bar{l}}(r_i^2-m^2)ip_{\mu}\mathscr{M}^{\mu\mu_1\dots\mu_k}(p,p_1,\dots,p_k,q_1,\dots,q_l,r_1,\dots,r_{\bar{l}}) \\ &=\sum_{i=1}^{l}\mathscr{M}^{\mu_1\dots\mu_k}(p_1,\dots,p_k,q_1,\dots,q_i+p,\dots,q_l,r_1,\dots,r_l) \\ &-\sum_{i=1}^{\bar{l}}\mathscr{M}^{\mu_1\dots\mu_k}(p_1,\dots,p_k,q_1,\dots,q_l,r_1,\dots,r_i+p,\dots,r_l)\,. \end{aligned} \tag{16.104}$$

In each summation, one pole is shifted from its on shell value, which means that the term vanishes when multiplied by appropriate $q_i^2 - m^2$ or $r_i^2 - m^2$, the right hand side of the equation vanishes so we have the Ward identity

$$p_\mu \mathscr{M}^\mu = 0\,. \tag{16.105}$$

Remark 16.6. *The above argument don't require* $p^2 = 0$*. Along similar lines, we can argue that replacing a photon propagator* $\Pi^{\mu\nu}(k)$ *by* $k^\mu k^\nu$ *gives zero.*

The path integral method shows his power. As a summary, we list several advantages of the path integral method

- In principle, non-pertubative definition of theory.
- Manifestly Lorentz invariant.
- Quantizing non-Abelian gauge symmetries.
- New perspective on classical (action) vs. quantum (integration measure).

Chapter 17

Renormalization

Tree diagram becomes disconnected upon cutting any internal lines, this observation follows from the fact that all momenta are linear combinations of external momenta. However, things are different in loop diagram. The loop diagrams are generically divergent due to the momentum flow in the internal lines. The strategy to deal with the divergence is

- parametrize the divergence (e.g. by introducing cut-off), isolated a characteristic finite piece (e.g. momentum space), this is the regularization scheme.

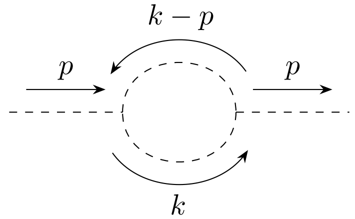
- absorb it by matching to a (finite) number of measured observables. This is the renormalization scheme.

  - In QED, these observables could be the mass and charge of the electron.

  More generally, we choose some observables at a fixed scale Λ and study the change of couplings as we vary Λ, this leads to the idea of renormalization group flow.

Example 17.1. *A typical example is ϕ^3 theory, with the Lagrangian $\mathscr{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \frac{g}{3!}\phi^3$, the loop diagram*

$$\mathscr{M}_{loop}(p) =$$



$$, \tag{17.1}$$

which is the building block of a bigger diagram. p is not necessarily on-shell. Based on the Feynman rules,

$$i\mathscr{M}_{loop}(p) = \frac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2 - m^2 + i\epsilon} \frac{i}{k^2 - m^2 + i\epsilon} \sim \int \frac{k^3 dk}{k^4} \sim \ln k \,, \tag{17.2}$$

where $\frac{1}{2}$ is the symmetric factor and the integral is logarithmic divergent. The UV interpretation of this divergence is that $\mathscr{L}$ is not valued at high energies. Here the estimation is rough, k^3 comes from the spherical coordinate Jacobian and k^2 estimated as the Euclidean innerproduct.

17.1 The Feynman parameter trick

Product of propagators integrated over a mutual momentum. We consolidate the momentum dependence by using the Feynman parameter. Notice that

$$\frac{1}{A \cdot B} = \int_0^1 dx \frac{1}{[A + (B - A)x]^2}\,, \tag{17.3}$$

where x is the Feynman parameter. Applying to the integral, we have

$$\frac{1}{A(k)B(k)} = \int dx \frac{1}{(k^2 + \ldots)^2}\,. \tag{17.4}$$

Example 17.2. *In the above loop diagram, we have $A = (k - p)^2 - m^2 + i\epsilon$, $B = k^2 - m^2 + i\epsilon$,*

$$\begin{aligned} \Rightarrow \quad A + (B - A)x &= (k - p)^2 - m^2 + \left(k^2 - (k - p)^2\right) x + i\epsilon \\ &= k^2 - 2pk + p^2 - m^2 + (2kp - p^2)x + i\epsilon \\ &= k^2 - -2kp(1 - x) + p^2(1 - x) - m^2 + i\epsilon \\ &= \left(k^2 - p(1 - x)\right)^2 - p^2(1 - x)^2 + p^2(1 - x) - m^2 + i\epsilon \\ &\xrightarrow{shift\,k} k^2 + p^2(1 - x)x - m^2 + i\epsilon = k^2 - \Delta + i\epsilon\,, \end{aligned} \tag{17.5}$$

where $\Delta = -(p^2(1 - x)x - m^2)$. Then the amplitude

$$i\mathscr{M}_{loop}(p) = \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{(k^2 - \Delta + i\epsilon)^2}\,. \tag{17.6}$$

17.2 Wick rotation

We want to replace k^2 by k_E^2, i.e. by Euclidean product. Consider

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^n}, \qquad n > 1\,. \tag{17.7}$$

Expand the denominator,

$$k^2 - \Delta + i\epsilon = \left[k^0 - \left(\sqrt{\vec{k}^2 + \Delta} - i\epsilon\right)\right] \left[k^0 + \left(\sqrt{\vec{k}^2 + \Delta} - i\epsilon\right)\right]\,. \tag{17.8}$$

Remark 17.1. *Here we redefine the infinitesimal parameter ϵ and take the infinitesimal advantage.*

Assume $\Delta > 0$ for concreteness, The integral contour (x-axis is the k^0), is as follows, The integral along the red contour is zero $\oint_C = 0$ because it contains no poles. The integral along the arc is also zero because the function drops off as $(k^0)^{-2n}$, then we have

$$\int_{-\infty}^{+\infty} dk^0 \int \frac{d^3k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^n} = \int_{-i\infty}^{+i\infty} dk^0 \int \frac{d^3k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^n}\,. \tag{17.9}$$

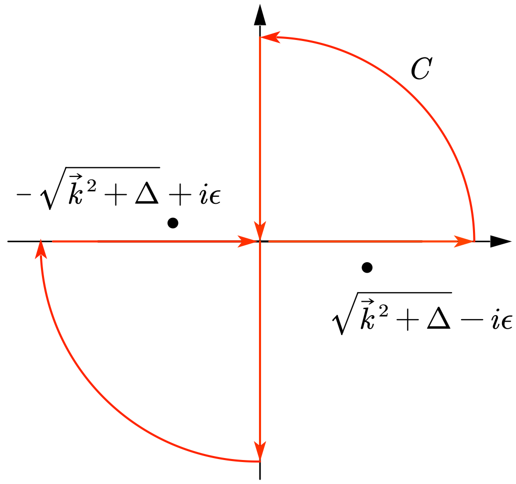


Figure 17.1: Integral contour

If we set $k^0 = ik_E^0$ (Wick rotation), the integral changes into

$$i\int_{-\infty}^{+\infty} dk_E^0 \int \frac{d^3k}{(2\pi)^4}\frac{1}{(-k_E^2-\Delta)^n}\,, \tag{17.10}$$

where $k_E^2 = (k_E^0)^2 + \vec{k}^2$.

Example 17.3. *If $n = 2$, the integration is*

$$\begin{aligned}
\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\epsilon)^2} &= i\int_{-\infty}^{+\infty}\frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+\Delta)^2} = i\int\frac{k^3 d\Omega_4 d_k}{(2\pi)^4}\frac{1}{(k^2+\Delta)^2}\\
&= i\frac{2\pi^2}{(2\pi)^4}\int_0^\infty \frac{\frac{1}{2}k^2dk^2}{(k^2+\Delta)^2} \overset{q^2=k^2+\Delta}{=} i\frac{1}{8\pi^2}\int_\Delta^\infty \frac{\frac{1}{2}(q^2-\Delta)dq^2}{q^4}\\
&= \frac{i}{8\pi^2}\lim_{c\to+\infty}\left[\ln q + \frac{1}{2}\frac{\Delta}{q^2}\right]_{q=\sqrt{\Delta}}^{q=\sqrt{c}}\\
&= \frac{i}{8\pi^2}\lim_{c\to+\infty}\left[\frac{1}{2}\ln c - \frac{1}{2}\ln\Delta - \frac{1}{2}\right]
\end{aligned} \tag{17.11}$$

where c is the upper bound and we drop the term $\frac{\Delta}{c}$ because c is a large number.

17.3 Hard cut-off and Pauli-Villars regularization

Definition 17.1. *Hard cut-off is the cut-off at high energy by introducing upper bound Λ*

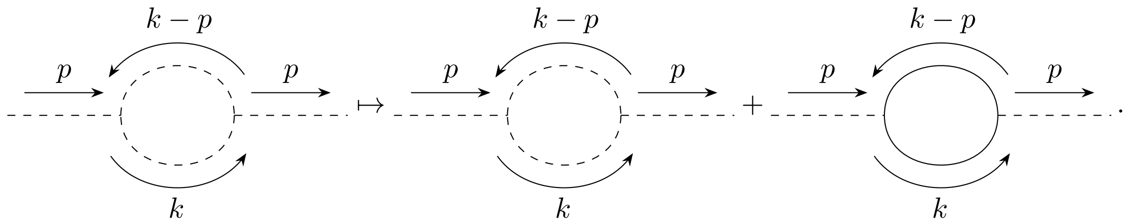
$$\int_0^\infty dk \to \int_0^\Lambda dk\,. \tag{17.12}$$

The advantage of this cut-off is that it is physically intuitive. The disadvantage is that it breaks the shift symmetry in momentum and clumsy to work with. The idea of Pauli-Villars regularization is to introduce fictitious (does not occur at in and out states) heavy (not produced in scattering, does not alter in low energy physics) particle to cancel the divergences, i.e. modifying UV of the theory.

Example 17.4. *For ϕ^3 theory, we introduce a fermionic scalar ψ, which will have a minus sign in loops diagram. It violated the spin-statistics. Such particles are called ghosts, which cannot appear as external states. The fermionic scalar couples to ϕ via $\phi\psi^2$ of mass $\Lambda \gg m$. The amplitude is then*

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^2} \mapsto \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^2} - \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Lambda^2 + i\epsilon)^2}\,. \tag{17.13}$$

Graphically speaking,



(17.14)

In the second integral, we replace $\Delta = \Lambda^2 - p^2x(1-x)$ by Λ^2 because Λ is much larger than p. Based on the equation (17.11), the result is

$$\frac{i}{16\pi^2} \lim_{c\to\infty} \left[\ln c - \ln \Delta - 1 - (\ln c - \ln \Lambda^2 - 1)\right] = -\frac{i}{16\pi^2} \ln \frac{\Delta}{\Lambda^2}\,. \tag{17.15}$$

Notice that the infinite part cancels because of the introduction of fictitious particle. Recovering the variable,

$$\begin{aligned} i\mathscr{M}_{loop}(p) = \frac{g^2}{2}\frac{-i}{16\pi^2}\int_0^1 dx \ln \frac{m^2 - p^2x(1-x)}{\Lambda^2} &\overset{m^2\sim 0}{=} -\frac{ig^2}{32\pi^2}\left(\int_0^1 dx \ln x(1-x) + \ln\frac{-p^2}{\Lambda^2}\right) \\ &= \frac{ig^2}{32\pi^2}\left[-2 - \ln\frac{-p^2}{\Lambda^2}\right] \end{aligned} \tag{17.16}$$

. A few remarks on this result.

- *By redefining Λ, we can absorb the constant term, which means that the -2 term is not physical.*
- *In the logarithm function, the number is minus, this is not reasonable because we use a not good approximation. But the point of the calculation is to see the disadvantage, we won't consider it seriously.*
- *The dependence $\ln(-p^2/\Lambda)$ on p is not altered by changing Λ as long as this change is p independent, which means that we decompose the UV and IR divergence.*

There are several disadvantages of Pauli-Villars regularization,

- Needs modification at higher loops.
- it is not a gauge invariant regulator, as PV ghost mush have the same Lorentz transformation properties/charges as particle. It must have large mass, which is incompatible with regularizing photon.

17.4 Dimensional regularization

This regularization scheme respects the gauge symmetry.

17.4.1 The idea

In fact, the integral

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta+i\epsilon)^2} < \infty \quad \text{for} \quad d<4\,. \tag{17.17}$$

It will logarithmically divergent at $d=4$. Inspired by this result, we define the real number dimension d, and evaluate the integral at $d=4-\epsilon$, where the ϵ plays the role $1/\Lambda$. The first thing we need to generalize is the integral measure

$$\int d^dk = \int \Omega_d \int dkk^{d-1}\,, \tag{17.18}$$

the k^{d-1} make senses for arbitrary $d\in\mathbb{R}$, but the Ω_d needs generalized definition. Mathematically we have

$$\left(\int_{-\infty}^{+\infty} dxe^{-x^2}\right)^d = \int d\Omega_d \int drr^{d-1}e^{-r^2} = \Omega_d\frac{1}{2}\Gamma\left(\frac{d}{2}\right)\,, \tag{17.19}$$

where the gamma function is defined as $\Gamma(z)=\int_0^\infty dxx^{z-1}e^{-x}$ for $\mathrm{Re}z>0$. We know that the result for the LHS is $(\sqrt{\pi})^d$. $\Gamma(n)=(n-1)!$ for $n\in\mathbb{N}$ and the negative integers and zero is simple poles of gamma function upon analytically continuation. Thus, the Ω_d for $d\in\mathbb{R}$ is

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\,. \tag{17.20}$$

17.4.2 A useful integral identity

$$\int_0^\infty dk\frac{k^a}{(k^2+\Delta)^b} = \Delta^{\frac{a+1}{2}-b}\frac{\Gamma(\frac{a+1}{2})\Gamma\left(b-\frac{a+1}{2}\right)}{2\Gamma(b)}\,, \tag{17.21}$$

where the equal holds in the mutual domain of convergence. Recall the integral we had before and keep the a,b,

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^{2a}}{(k^2-\Delta)^b} = i\int\frac{d^dk_E}{(2\pi)^d}\frac{(-k_E^2)^a}{(k^2+\Delta)^b} = i(-1)^{a-b}\frac{2\pi^{d/2}}{(2\pi)^d}\frac{1}{\Delta^{b-a-d/2}}\frac{\Gamma\left(a+\frac{d}{2}\right)\Gamma\left(b-a-\frac{d}{2}\right)}{2\Gamma\left(b\right)\Gamma\left(\frac{d}{2}\right)}\,. \tag{17.22}$$

17.4.3 The coupling constant dimension in dimensional regularization

To obtain logs only of dimensionless quantities, we need to keep track of the mass dimension of our coupling constants. Determine mass dimension of field by containing with kinetic terms and retaining $[S]=0$.

$$[\psi]=\frac{d-1}{2},\quad [A^\mu]=\frac{d-2}{2},\quad d=[e\bar{\psi}\not{A}\psi]=[e]+(d-1)+\frac{d}{2}-1 \quad\Rightarrow\quad [e]=\frac{4-d}{2}\,. \tag{17.23}$$

In $d=4$, the charge is dimensionless. We want to keep e dimensionless in d dimension, we introduce a scalar μ such that

$$e\to e\mu^{\frac{4-d}{2}}\,. \tag{17.24}$$

We replace the coupling constant which e appears in loops by $e\mu^{\frac{4-d}{2}}$ to avoid the logs of dimensionful quantities. And we maintain e everywhere. The final results shouldn't depend on μ.

17.4.4 Logarithmically divergent integrals in dimension regulator

$$\int \frac{d^dk}{(2\pi)^d}\frac{e^2\mu^{4-d}}{(k^2-\Delta+i\epsilon)} \overset{a=0,b=2}{=} \mu^{4-d}e^2 i(-1)^2\frac{1}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-d/2}}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)}{\Gamma\left(2\right)\Gamma\left(\frac{d}{2}\right)}\,, \tag{17.25}$$

which diverges at $d=4$. Set $d=4-\epsilon$ where ϵ is small, using the expansion of Gamma function

$$\Gamma\left(\epsilon\right) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\,, \tag{17.26}$$

where γ_E is the Euler-Mascheroni constant. This gives

$$\mu^{4-d}\frac{ie^2}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\,, \tag{17.27}$$

using the expansion $(4\pi)^{d/2} = (4\pi)^{2-\epsilon/2} = (4\pi)^2(4\pi)^{-\epsilon/2} \approx (4\pi)^2(1+\frac{\epsilon}{2}\ln 4\pi)$, $\mu^\epsilon \approx 1+\epsilon\ln\mu$, $\Gamma\left(\frac{\epsilon}{2}\right) \approx \frac{2}{\epsilon} - \gamma_E$, $\Delta^{-\epsilon/2} \approx 1 - \frac{\epsilon}{2}\ln\Delta$, we have

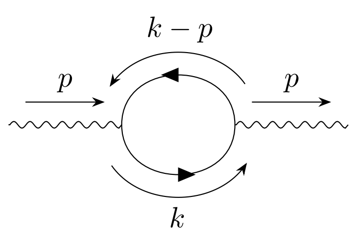
$$\begin{aligned} \Rightarrow &= \frac{ie^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + (-\gamma_E + \ln\mu^2 + \ln 4\pi - \ln\Delta) + \mathcal{O}(\epsilon)\right] \\ &= \frac{ie^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \ln\left(4\pi e^{-\gamma_E}\mu^2\right) + \mathcal{O}(\epsilon)\right] = \frac{ie^2}{(4\pi)^2}\left[\frac{2}{\epsilon} + \ln\tilde{\mu}^2 + \mathcal{O}(\epsilon)\right]\,, \end{aligned} \tag{17.28}$$

where $\tilde{\mu}^2 = \left(4\pi e^{-\gamma_E}\mu^2\right)$.

17.5 Renormalizing at the level of the S -matrix

Example 17.5. *Vacuum polarization. In QED, a typical loop diagram is*

$$i\Pi_2^{\mu\nu} =$$

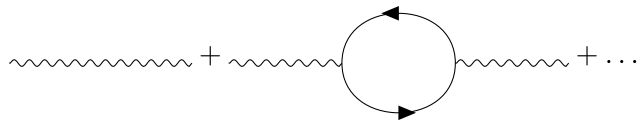


$$\begin{aligned} &= -(-ie)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k-p)^2-m^2+i\epsilon}\frac{i}{k^2-m^2+i\epsilon}\,\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{p}+m)\gamma^\nu(\not{k}+m)\right] \\ &= i\left[\Delta_1(p^2,m^2)p^2 g_{\mu\nu} + \Delta_2(p^2,m^2)p^\mu p^\nu\right]\,, \end{aligned} \tag{17.29}$$

with external legs amputated (and not replaced by polarization tensors). The subscript 2 means it is order 2 in coupling. Δ_1 *and* Δ_2 *are called the form factors, which are functions of momenta (and masses) left to determine once the kinetics has been taken into account (for non pertubative processes, e.g. in QCD, determined via measurement).* $\Pi_2^{\mu\nu}$ *contributes to the dressed photon propagator*

$$\langle\Omega|\,\mathscr{T}\left\{A^\mu(x)A^\nu(y)\right\}|\Omega\rangle = \int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}iG^{\mu\nu}(p)\,. \tag{17.30}$$

with

$iG^{\mu\nu}(p) =$ 

$$
= \frac{-ig_{\mu\nu}}{p^2+i\epsilon} + \frac{-ig^{\mu\rho}}{p^2+i\epsilon} i(\Pi_2)_{\rho\sigma} \frac{-ig^{\sigma\nu}}{p^2+i\epsilon} + \dots\,, \tag{17.31}
$$

where we have already chosen the Feynman gauge. Based on the expression of $\Pi_2^{\mu\nu}$, *we have*

$$
iG^{\mu\nu}(p) = -i\frac{(1+\Delta_1)g^{\mu\nu} + \Delta_2\frac{p^\mu p^\nu}{p^2}}{p^2+i\epsilon}\,. \tag{17.32}
$$

Recall by Ward identity, any S-matrix in which $G^{\mu\nu}$ *will be inserted will be dependent of* Δ_2*-term. As* $p_\mu \mathscr{M}^\mu = 0$, *we can drop the term proportional to* $p^\mu p^\nu$. *Note further, by considering the* $\gamma \to \gamma$ *S-matrix, the Ward identity will give*

$$
\begin{aligned}
p_\mu \Pi_2^{\mu\nu} = 0 \quad &\Rightarrow \quad p_\mu\left[\Delta_1(p^2,m^2)p^2 g_{\mu\nu} + \Delta_2(p^2,m^2)p^\mu p^\nu\right] \overset{!}{=} 0 \\
&\Rightarrow \Delta_1 p^2 p^\nu + \Delta_2 p^2 p^\nu \overset{!}{=} 0 \quad \Rightarrow \quad \Delta_1 = -\Delta_2\,.
\end{aligned} \tag{17.33}
$$

We can check it by explicit calculation.

17.5.1 Regularizing the divergence

Following the route about calculating the integral (17.29) (exercise)

- evaluate the trace,
- introducing Feynman parameters,
- dropped term proportional to $p^\mu p^\nu$,
- performing dimensional regularization,

we have

$$
i\Pi_2^{\mu\nu}(p^2) = i(-p^2 g^{\mu\nu})e^2\Pi_2(p^2) + p^\mu p^\nu\text{-terms}\,, \tag{17.34}
$$

where

$$
\Pi_2(p^2) = \frac{1}{2\pi^2}\int_0^1 dx x(1-x)\left[\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{m^2 - p^2 x(1-x)}\right] \tag{17.35}
$$

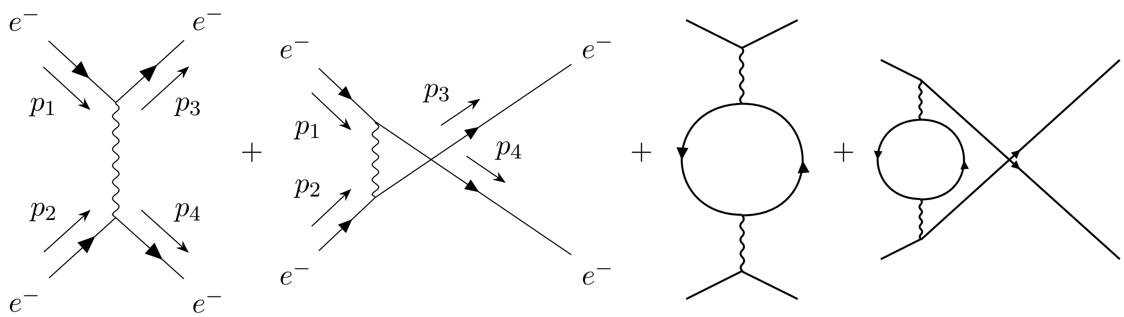
with $\tilde{\mu}^2 = 4\pi e^{-\gamma_E}\mu$.

17.5.2 Renormalizing vacuum polarization by computing 2-2 scattering to experiment

The dressed propagator

$$
iG^{\mu\nu} = -i\frac{1 - e^2\Pi_2(p^2)}{p^2+i\epsilon} g^{\mu\nu} + p^\mu p^\nu\text{-terms} + \mathcal{O}(e^4)\,, \tag{17.36}
$$

arises in $e^-e^- \to e^-e^-$ scattering,

$i\mathscr{M} =$ 

(17.37)

$$= (-ie)^2 \bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma^\nu u(p_2) iG^{\mu\nu} - (p_3 \leftrightarrow p_4)\,,$$

where the minus sign comes from exchanging the fermion lines. Note that we can check explicitly that $p^\mu p^\nu$-term does not contribute. We claim that the loop correction can be interpreted as correction to the classical potential. To see this, take non-relativistic limit $m \gg 1$, the polarization tensor is

$$\begin{aligned} u(p) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} \quad \text{with} \quad \sigma = (\mathbb{1}, \vec{\sigma}),\ \bar{\sigma} = (\mathbb{1}, -\vec{\sigma}),\ \sigma = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \\ &\Rightarrow u(p) = \sqrt{p^0} \begin{pmatrix} \sqrt{\mathbb{1} - \vec{p}\cdot\vec{\sigma}/p^0}\xi \\ \sqrt{\mathbb{1} + \vec{p}\cdot\vec{\sigma}/p^0}\xi \end{pmatrix}, \end{aligned} \tag{17.38}$$

where $p^0 = \sqrt{m^2 + \vec{p}^2}$. Take the non-relativistic limit,

$$\Rightarrow \quad \bar{u}(p_3)\gamma^\mu u(p_1) = \begin{cases} \mu = 0 \quad u^\dagger(p_3)u(p_1) \approx m \begin{pmatrix} \xi_3^\dagger & \xi_3^\dagger \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = 2m\xi_3^\dagger \xi_1 \\ \mu = i \quad u^\dagger(p_3)\gamma^0\gamma^i u(p_1) = u^\dagger(p_3) \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} u(p_1) \approx 0 \quad \text{(leading order)}. \end{cases} \tag{17.39}$$

$$\Rightarrow \quad i\mathscr{M}_t = -e^2 \left[-i\frac{1 - e^2\Pi_2(p^2)}{p^2} g^{00} \right] 4m^2 \xi_3^\dagger \xi_1 \xi_4^\dagger \xi_2 = -e^2 \left[-i\frac{1 - e^2\Pi_2(p^2)}{p^2} g^{00} \right] 4m^2 \delta_{\sigma_1\sigma_3}\delta_{\xi_2\xi_4}\,, \tag{17.40}$$

and $p = p_3 - p_1$, as $p_i^0 \sim m$, we have $p^2 = -\vec{p}^2$. Let's compare to non-relativistic Bonn approximation

$$S_{\beta\alpha}^{\text{Bonn}} \delta(\beta - \alpha) - 2\pi i \delta(E_\alpha - E_\beta) \langle \phi_\beta | V | \phi_\alpha \rangle\,, \tag{17.41}$$

where $T_{\beta\alpha} = \langle \phi_\beta | V | \phi_\alpha \rangle$, V is the interacting Hamiltonian $H' = H + V$ and $|\phi_\alpha\rangle$ is the free state. Applied to $2 \to 2$ scattering, in local, spin-independent central potential,

$$T\left(\phi_{p_1,\sigma_1} + \phi_{p_2,\sigma_2} \to \phi_{p_3,\sigma_3} + \phi_{p_4,\sigma_4}\right) = {}_A\langle \Phi_3, \Phi_4 | V(|\hat{\vec{x}} - \hat{\vec{y}}|) | \Phi_1, \Phi_2 \rangle_A\,. \tag{17.42}$$

A denotes the anti-symmetrization. Expanding the integral,

$$\begin{aligned} = \int d^3\vec{x}_1 d^3\vec{x}_2 d^3\vec{x}_3 d^3\vec{x}_4 \langle \vec{p}_3, \vec{p}_4 | \vec{x}_3, \vec{x}_4 \rangle \langle \vec{x}_3, \vec{x}_4 | V(|\vec{x}_1 - \vec{x}_2|) | \vec{x}_1, \vec{x}_2 \rangle \langle \vec{x}_1, \vec{x}_2 | \vec{p}_1, \vec{p}_2 \rangle \delta_{\sigma_1\sigma_2}\delta_{\sigma_2\sigma_4} \\ -(p_3, \sigma_3 \leftrightarrow p_4, \sigma_4)\,, \end{aligned} \tag{17.43}$$

the last term means that exchanging the position of the quantity relates to the particle 3 and 4. Notice that

$$\langle \vec{p}_3, \vec{p}_4 | \vec{x}_3, \vec{x}_4 \rangle = \left(\frac{1}{(2\pi)^{3/2}} \right)^2 e^{-i(\vec{p}_3 \cdot \vec{x}_3 + \vec{p}_4 \cdot \vec{x}_4)} \,, \tag{17.44}$$

we have

$$\begin{aligned} &= \frac{1}{(2\pi)^6} \int d^3x_1 d^3x_2 e^{i(\vec{p}_1 - \vec{p}_3)\cdot \vec{x}_1 + i(\vec{p}_2 - \vec{p}_4)\cdot \vec{x}_2} V\left(|\vec{x}_1 - \vec{x}_2|\right) \delta_{\sigma_1\sigma_3}\delta_{\sigma_2\delta_4} - (3 \leftrightarrow 4) \\ &\overset{\vec{x}_1 = \vec{x}_2 + \vec{r}}{=} \frac{1}{(2\pi)^6} (2\pi)^3 \int d^3r V(|\vec{r}|) e^{i(\vec{p}_1 - \vec{p}_3)\cdot \vec{r}} \delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} - (3 \leftrightarrow 4) \,. \end{aligned} \tag{17.45}$$

With $V(|\vec{r}|) = \frac{e^2}{|\vec{r}|}$, $\tilde{V}(\vec{p}) = 4\pi \frac{e^2}{|\vec{p}|^2}$, upon matching normalization

$$\tilde{V}(p^2) = \frac{e^2}{p^2}\left(1 - e^2\Pi_2(p^2) + \dots\right) \,, \tag{17.46}$$

QFT calculation indeed reproduces classical Coulomb potential. The experiment will measure $\tilde{V}(p^2)$ at some reference scale p^0, it is no doubt that $\tilde{V}(p_0^2)$ is finite, but why $\Pi_2(p_0^2)$ is infinite? Reconsider the identification of e in Lagrangian with the measured electric charge of electron. We define the renormalized charge e_R at reference scale p^0.

$$\tilde{V}(p_0^2) = \frac{e_R^2}{p_0^2} \,, \tag{17.47}$$

which is called the renormalization condition. Combine with the relation (17.46), we have

$$e_R^2 = p_0^2 \tilde{V}(p_0^2) = e^2 - e^4 \Pi_2(p_0^2) + \dots \,. \tag{17.48}$$

The left hand side should be finite. Then comes to the renormalization step: replace e-dependence (infinite, non-measurable) by e_R-dependence (finite, measurable). Invert the equation (17.48) as formal power series, we have

$$e^2 = e_R^2 + e_R^2 \Pi_2(p_0^2) + \dots \tag{17.49}$$

and substitute with p_0 as arbitrary momentum,

$$p^2 \tilde{V}(p^2) = e^2 - e^4 \Pi_2(p^2) + \dots = e_R^2 + e_R^4 \Pi_2(p_0^2) - e_R^4 \Pi_2(p^2) + \mathcal{O}(e_R^6) \,. \tag{17.50}$$

Let us choose $p_0 \to 0$ (this corresponds to measuring charge at ∞), and based on the equation (17.35), we have

$$\Rightarrow \quad \Pi_2(p^2) - \Pi_2(0) = -\frac{1}{2\pi^2} \int_0^1 x(1-x) \ln\left(1 - \frac{p^2}{m^2} x(1-x)\right) \,, \tag{17.51}$$

where the ϵ and $\tilde{\mu}$ dependence cancels. It leads to the momentum dependent correction to potential.

17.5.3 Small momentum approximation and the Lamb shift

For $|p^2| \ll m^2$, we truncate the logarithm in at first order,

$$\Pi_2(p^2) - \Pi_2(0) = -\frac{1}{2\pi^2} \int_0^1 x(1-x) \left[-\frac{p^2}{m^2} x(1-x) \right] = \frac{p^2}{60\pi^2 m^2} \,, \tag{17.52}$$

$$\Rightarrow \quad \tilde{V}(p^2) = \frac{e_R^2}{p^2} - \frac{e_R^4}{60\pi^2 m^2} + \dots . \tag{17.53}$$

Inverse Fourier transform the expression we get the correction of the potential,

$$V(r) = \frac{e_R^2}{4\pi r} - \frac{e_R^4}{60\pi^2 m^2}\delta(r) + \dots , \tag{17.54}$$

where the second term (first order correction) is called the Uehiling term. The correction has support at origin. Hence, it affects the s ($l = 0$) orbit but not the p ($l = 1$) orbit. It destroys the degeneracy between $2s$ and $2p$ level, as measured by Lamb. This is a contribution called the Lamb shift. The full treatment requires external field method.

17.5.4 The running coupling, screening, and the Landau pole

Consider $Q^2 = -p^2 \gg m$ (Again, p is not on-shell), this gives

$$\tilde{V}(Q^2) = -\frac{e_R^2}{Q^2} - \frac{e_R^4}{Q^2}\frac{1}{2\pi^2}\int_0^1 dx x(1-x)\ln\left[1 + \frac{Q^2}{m^2}x(1-x)\right] . \tag{17.55}$$

The logarithm

$$\ln\left[1 + \frac{Q^2}{m^2}x(1-x)\right] \sim \ln\left[\frac{Q^2}{m^2}x(1-x)\right] \sim \ln\frac{Q^2}{m^2} + \ln x(1-x) \sim \ln\frac{Q^2}{m^2} , \tag{17.56}$$

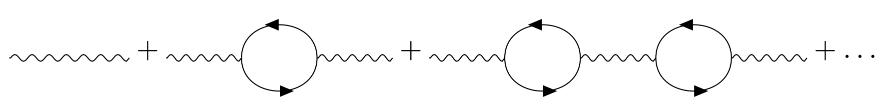
then the integral reduces to

$$\begin{aligned} &\approx -\frac{e_R^2}{Q^2}\left(1 + \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2} + \dots\right) \\ &= -\frac{e_{ff}^2}{Q^2} , \end{aligned} \tag{17.57}$$

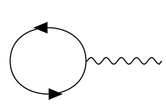
with $e_{ff}^2 = e_R^2\left(1 + \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2} + \dots\right)$. The effective charge is Q-dependent, which is called the running coupling. There are two limits needs attention

- $Q^2 \to m^2$, which is equivalent to increasing the distance, the effective charge decreases. The interpretation is that the creation of virtual electron-positrons pairs that screen the charge.

- Q^2 is large. The effective charge diverges because of the term $e_R^2 \ln\frac{Q^2}{m^2}$. Now we introduce Bettes estimate. First, each loops will have

  $iG^{\mu\nu} =$  (17.58)

  Notice that each unit

   $= \left[-ip^2 g_{\rho\sigma} e_R^2 \left(\Pi_2(p^2) - \Pi_2(0)\right)\right]\left(-\frac{ig^{\sigma\nu}}{p^2}\right) ,$ (17.59)

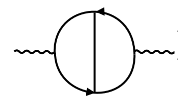
with $\Pi_2(p^2) - \Pi_2(0) = -\frac{1}{12\pi^2}\ln\frac{Q^2}{m^2}$, we have

$$\left[ip^2 g_{\rho\sigma} e_R^2 \frac{1}{12\pi^2}\ln\frac{Q^2}{m^2}\right]\left(-\frac{ig^{\sigma\nu}}{p^2}\right) = \delta^\nu_\rho \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}\,. \tag{17.60}$$

Then the expression (17.58) generates

$$iG^{\mu\nu} = \left(-\frac{ig^{\mu\nu}}{p^2}\right) + \left(-\frac{ig^{\mu\nu}}{p^2}\right)\frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2} + \left(-\frac{ig^{\mu\nu}}{p^2}\right)\left(\frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}\right)^2 + \ldots \tag{17.61}$$

$$\begin{aligned}\Rightarrow \quad e_{ff}^2(Q^2) &= e_R^2\left[1 + \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2} + \left(\frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}\right)^2 + \ldots\right] \\ &= \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}} + \ldots\,,\end{aligned} \tag{17.62}$$

where the ... represents some diagrams are not included by this geometric summation (e.g. ).

The result has pole at $\ln\frac{Q^2}{m^2} = \frac{12\pi^2}{e_R^2}$. This breaks down the perturbation theory of ϕ^3, called the Landau pole.

17.6 Renormalizing Green's function

Rather than studying observables (S-matrix elements) directly, we will study the building blocks, which are the n-point functions, organized by number n of insertions.

One point function

Because the vacuum is the translational invariant, the one point function is

$$\langle\Omega| A^\mu(x) |\Omega\rangle = \langle\Omega| e^{ipx} A^\mu(x) e^{-ipx} |\Omega\rangle = \langle\Omega| A^\mu(x) |\Omega\rangle\,. \tag{17.63}$$

The VEVs are constant, set by boundary conditions on your theory. As for the Lorentz transformation,

$$\langle\Omega| U(\Lambda) A^\mu(0) U^{-1}(\Lambda) |\Omega\rangle = \Lambda^\mu{}_\nu \langle\Omega| A^\nu(0) |\Omega\rangle\,, \tag{17.64}$$

by Lorentz invariant of the vacuum $U(\Lambda)|\Omega\rangle = |\Omega\rangle$, the left hand side is always $\langle\Omega| A^\mu(0) |\Omega\rangle$, thus $\langle\Omega| A^\mu(0) |\Omega\rangle = \Lambda^\mu{}_\nu \langle\Omega| A^\nu(0) |\Omega\rangle$ for all Λ, the only possibility is that $\langle\Omega| A^\mu(0) |\Omega\rangle \equiv 0$. Only scalar fields can have non-vanishing VEVs without breaking Lorentz invariance. This conclusion will play an important role in the Standard model and the context of spontaneous symmetry breaking, but not in QED.

Two point functions

In QED, there are four types of the two point functions,

- $\langle\Omega| \mathscr{T}\{A^\mu(x) A^\nu(y)\} |\Omega\rangle$ will leads to the vacuum polarization, which we discussed above.

- $\langle\Omega| A^\mu(x)\psi(y) |\Omega\rangle = \langle\Omega| \mathscr{T}\left\{A^\mu\bar{\psi}(y)\right\} |\Omega\rangle = 0$ as no possible Feynman diagram.
- $\langle\Omega| \psi(x)\bar{\psi}(y) |\Omega\rangle$ will leads to the problem called the electron self energy.

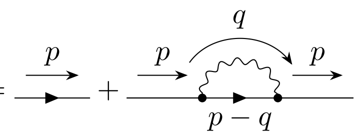
All other combination of two spinors vanish, by same argument as above.

$$\langle\Omega| \mathscr{T}\left\{\psi(x)\bar{\psi}(y)\right\} |\Omega\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} iG(\not{p})\,. \tag{17.65}$$

Remark 17.2. $G = G(p^2, \not{p})$. *Because* $p^2 = \not{p}\not{p}$, *so we just write* $G(\not{p})$.

Consider the Feynman diagram up to order 2, and define $iG_0(\not{p}) = \frac{i}{\not{p}-m}$,

$$iG(\not{p}) = \tag{17.66}$$



$$= iG_0(\not{p}) + iG_0(\not{p})(i\Sigma_2(\not{p}))iG_0(\not{p})\,.$$

where the subscript 2 denote that it is the second order contribution, In Feynman gauge, we have

$$i\Sigma_2(\not{p}) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^\nu \frac{-ig_{\mu\nu}}{(p-k)^2+i\epsilon}\,. \tag{17.67}$$

Following the same steps of estimation, the UV divergent part of Σ_2 is

$$[i\Sigma_2(\not{p})]_{\text{UV divergent}} = \frac{\alpha}{\pi}\frac{\not{p}-4m}{2\epsilon}\,, \quad \alpha = \frac{e^2}{4\pi}\,. \tag{17.68}$$

Remark 17.3. *Unlike* Π_2, $[\Sigma_2]_{divergent}$ *has two terms with different momentum dependence, which means that we will require two renormalization conditions.*

17.6.1 Mass and field renormalization

Recall Π_2 divergence was absorbed in replacing charge e with renormalized charge e_R. We absorb $[\Sigma_2]_{\text{divergent}}$ by replacing

$$m_0 \to m_R\,, \quad \psi^0 \to \psi^R\,. \tag{17.69}$$

We set

$$\begin{aligned} m_0 &= Z_m m_R = (1+\delta_m)m_R\,, \quad \delta_m = \mathcal{O}(e_R^2) \\ \psi^0(x) &= \sqrt{Z_2}\psi^R(x) = \sqrt{1+\delta_2}\psi^R(x)\,, \quad \delta_2 = \mathcal{O}(e_R^2)\,. \end{aligned} \tag{17.70}$$

δ_m, δ_2 are formal power series in e_R, beginning at order e_R^2. We quantify the variable by using LSZ formula.

$$\langle\Omega| \mathscr{T}\left\{\psi^R(x)\bar{\psi}^R(y)\right\} |\Omega\rangle = \frac{1}{Z_2}\langle\Omega| \mathscr{T}\left\{\psi^0(x)\bar{\psi}^0(y)\right\} |\Omega\rangle\,, \tag{17.71}$$

at tree level

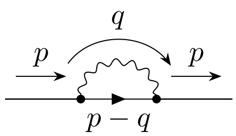
$$\frac{1}{Z_2}\frac{i}{\not{p}-m_0} = \frac{1}{1+\delta_2}\frac{i}{\not{p}-m_R-\delta_m m_R} = (1-\delta_2)\frac{i}{(\not{p}-m_R)\left(1-\frac{\delta_m m_R}{\not{p}-m_R}\right)} + \mathcal{O}(e_R^4)\,, \tag{17.72}$$

where we expand the denominator $(1+\delta_2)^{-1} = 1-\delta_2+\mathcal{O}(e_R^4)$. Using the same trick again, the equation becomes

$$\begin{aligned}
\frac{1}{Z_2}\frac{i}{\not p - m_0} &= (1-\delta_2)\frac{i}{\not p - m_R}\frac{\not p - m_R + \delta_m m_R}{\not p - m_R} + \mathcal{O}(e_R^4) \\
&= \frac{i}{\not p - m_R} + \frac{i}{\not p - m_R}\left[-i(-\delta_2 \not p + \delta_2 m_R + \delta_m m_R)\right]\frac{i}{\not p - m_R} + \mathcal{O}(e_R^4) \\
&= \frac{i}{\not p - m_R} + \frac{i}{\not p - m_R}\left[i(\delta_2 \not p - (\delta_2+\delta_m) m_R)\right]\frac{i}{\not p - m_R} + \mathcal{O}(e_R^4)\,,
\end{aligned} \tag{17.73}$$

which the right structure to absorb the divergence in Σ_2. Substitute the above result (the modification to $iG_0(\not p)$) into the equation (17.66) and collect the result up to the order e_R^2,

$$iG^R(\not p) = \frac{i}{\not p - m_R} + \frac{i}{\not p - m_R}\left[i(\delta_2 \not p - (\delta_2+\delta_m)m_R) + i\Sigma_2(\not p)\right]\frac{i}{\not p - m_R} + \mathcal{O}(e_R^4)\,. \tag{17.74}$$

The divergent part is $[i\Sigma_2(\not p)]_{\text{UV divergent}} = \frac{\alpha}{\pi}\frac{\not p - 4m}{2\epsilon}$, where we only keep the lowest order term. Replacing m_0 by m_R in the graph 

modifies the result at order e_R^4. We can absorb the divergence e.g. by defining

$$\begin{aligned}
\delta_2 &= -\frac{\alpha}{4\pi}\frac{2}{\epsilon}\,, \\
\delta_2 + \delta_m &= -\frac{\alpha}{\pi}\frac{2}{\epsilon} \\
\Rightarrow \quad \delta_m &= -\frac{3\alpha}{4\pi}\frac{2}{\epsilon}.
\end{aligned} \tag{17.75}$$

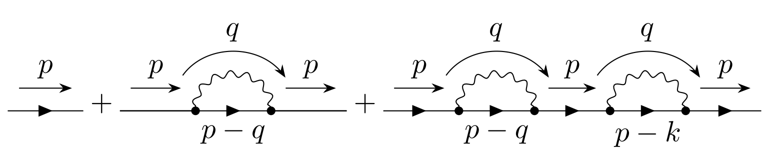
Here α has e-dependence. Since we only consider up to the order e_R^2, we can replace e in the lowest order $\alpha = \frac{e_R^2}{4\pi}$.

17.6.2 Subtraction scheme MS, $\overline{\text{MS}}$, on-shell subtraction

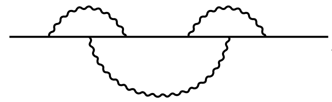
Modifying δ_2, δ_m by any finite contribution still absorbs the infinities, which give rise to equally admissible renormalization scheme.

- Minimal Subtraction (MS): define δ with no finite part as equation (17.75).
- Modified Minimal Subtraction ($\overline{\text{MS}}$), define δ that subtracts the part $\frac{2}{\epsilon} + \ln 4\pi - \gamma_E$.
- On-shell subtraction.

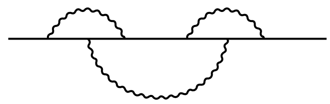
We introduce the on-shell subtraction scheme in detail. We want to identify m_R with the physical mass, defined as the location of the pole of the dressed propagator. As for vacuum polarization, consider

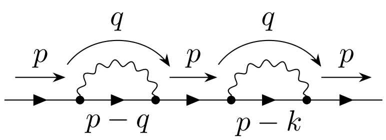
$G(\not p) =$ 

, (17.76)

which is the geometric sum, but does not encompass, e.g. . To remedy this, we define 1-particle irreducible (1 PI) diagrams:

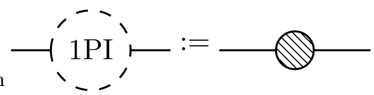
Definition 17.2. *1-particle irreducible diagrams (1PI): diagrams can not be divided into two non-trivial diagrams by cutting a single line.*

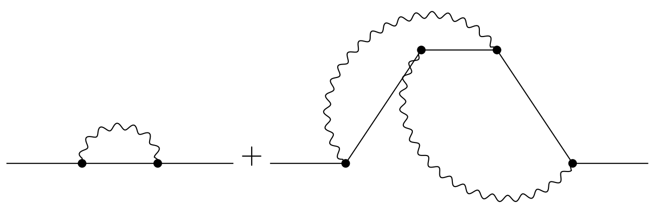
Example 17.6. *For example, the diagram*  *is 1PI.*

The diagram 

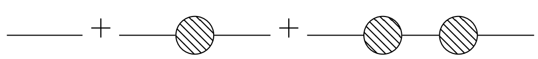
is not 1PI.

Then we define

$$i\Sigma(\not{p}) = \sum_{\text{1PI diagram}}$$ 

=  + ... (17.77)

where the external legs are amputated. Then

$$iG(\not{p}) =$$  + ...

$$\begin{aligned} &= \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m} i\Sigma \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m} i\Sigma \frac{i}{\not{p}-m} i\Sigma \frac{i}{\not{p}-m} + \dots \\ &= \frac{i}{\not{p}-m} \sum_{n=0}^{\infty} \left(i\Sigma \frac{i}{\not{p}-m} \right)^n = \frac{i}{\not{p}-m} \frac{1}{1 - i\Sigma \frac{i}{\not{p}-m}} = \frac{i}{\not{p}-m+\Sigma(\not{p})} \,. \end{aligned} \tag{17.78}$$

The quantum corrections shift position of pole of complete electron propagator away from free-value, which leads to the modification of LSZ as we required. The complete propagator for renormalized fields is then

$$iG^R(\not{p}) = \frac{1}{1+\delta_2} \frac{i}{\not{p}-m_0+\Sigma(\not{p})} = \frac{i}{\not{p}-m_R+\Sigma_R(\not{p})} \,, \tag{17.79}$$

with $\Sigma_R(\not{p})$ defined by the above equation. Expand to the order e_R^2, we have

$$\Sigma_R(\not{p}) = \Sigma_2(\not{p}) + \delta_2 \not{p} - (\delta_m + \delta_2) m_R + \mathcal{O}(e_R^4) \,. \tag{17.80}$$

The position of the pole of the complete propagator is a good definition of physical mass (pole mass). The renormalized propagator should have a single pole at $\not{p} = m_P$ with residue i.

Remark 17.4. *This pole mass is physical and independent of any subtraction scheme.*

From Eq. (17.79), for $G^R(\not{p})$ to have a pole at $\not{p} = m_P$, it must satisfy

$$m_P - m_R + \Sigma_R(m_P) = 0\,. \tag{17.81}$$

This is the condition defines the pole mass, which is independent of the choice of the subtraction scheme. The Renormalization conditions in on-shell renormalization scheme are

- $m_P - m_R = 0$, i.e. choose renormalized mass equal to physical mass.

- fix residue at m_P to be i (convenient for LSZ).

The first step gives

$$\begin{aligned}\Sigma_R(m_P) &= \Sigma_2(m_P) + \delta_2 m_P - (\delta_m + \delta_2) m_P|_{m_R = m_P} + \ldots \\ &= \Sigma_2(m_P) - \delta_m m_P \stackrel{!}{=} 0\,.\end{aligned} \tag{17.82}$$

The second step gives

$$i = \lim_{\not{p} \to m_P} (\not{p} - m_P) \frac{i}{\not{p} - m_R + \Sigma_R(\not{p})}\,, \tag{17.83}$$

expanding $\Sigma_R(\not{p})$ around m_P,

$$\begin{aligned}\Sigma_R(\not{p}) &= \Sigma_R(m_P) + (\not{p} - m_P)\, \frac{d}{d\not{p}} \Sigma_R(\not{p})\bigg|_{\not{p} = m_P} + \ldots \\ &= m_R - m_P + (\not{p} - m_P)\, \frac{d}{d\not{p}} \Sigma_R(\not{p})\bigg|_{\not{p} = m_P} + \ldots\,.\end{aligned} \tag{17.84}$$

Then the residue becomes

$$\lim_{\not{p} \to m_P} \frac{i}{1 + \frac{d}{d\not{p}} \Sigma_R(\not{p})\Big|_{\not{p} = m_P}} = i \quad \Rightarrow \quad \frac{d}{d\not{p}} \Sigma_R(\not{p})\bigg|_{\not{p} = m_P} \stackrel{!}{=} 0\,. \tag{17.85}$$

We will show that

$$\begin{aligned}\delta_m &= \frac{\alpha}{2\pi} \left(-\frac{2}{\epsilon} - \frac{3}{2} \ln \frac{\tilde{\mu}^2}{m_P^2} - 2 \right) \\ \delta_2 &= -\frac{\alpha}{2\pi} \left(\frac{1}{\epsilon} + \frac{1}{2} \ln \frac{\tilde{\mu}}{m_P^2} + 2 + \ln \frac{m_\gamma^2}{m_P^2} \right)\,,\end{aligned} \tag{17.86}$$

where m_γ is the IR regulator. To justify manipulation, we set

$$\begin{aligned}&\Sigma_R(\not{p}) = \alpha \not{p} + \beta \\ \Rightarrow \frac{i}{\not{p} - m_R + \Sigma_R(\not{p})} &= \frac{i}{(\alpha + 1)\not{p} - m_R + \beta} = \frac{i\left[(\alpha + 1)\not{p} + m_R - \beta\right]}{(\alpha + 1)^2 p^2 - (m_R - \beta)^2}\,,\end{aligned} \tag{17.87}$$

where the pole is

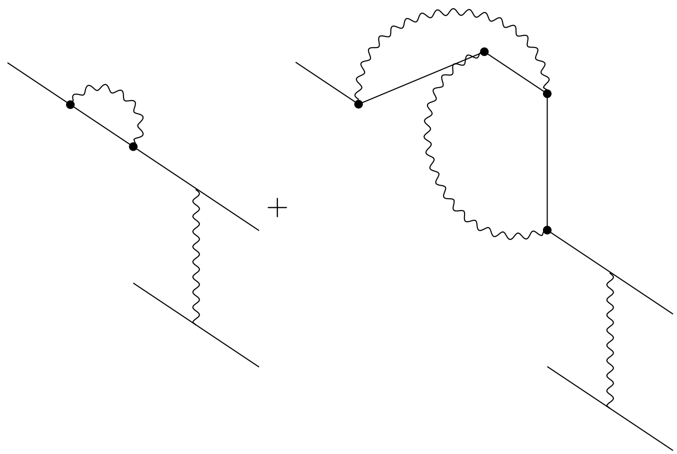
$$p^2 = \frac{(m_R - \beta)^2}{(\alpha + 1)^2} = m_P^2\,. \tag{17.88}$$

Also, from the equation (17.81), we have

$$\alpha m_P + \beta = \Sigma_R(m_R) = m_R - m_P \quad \Rightarrow \quad m_P = \frac{m_R - \beta}{\alpha + 1}\,. \tag{17.89}$$

17.6.3 LSZ and the shift of the pole of the complete electron propagator

LSZ formula tells us that the residue of the appropriate Green's function at $p_{\text{ext}}^2 = m^2$, where m is the physical mass, entered via dispersion relation $p^2 = m^2$. We need to replace m_0 by m_P in our considerations and multiply the time-ordered product by $\lim_{p^2 \to m_P^2}(-p^2 + m_P^2)$ to amputates the outermost propagator, i.e. all diagrams



$$+ \qquad\qquad + \ldots \tag{17.90}$$

just serve to shift the poles from m_0 to m_P in perturbation theory. LSZ prescription: To obtain S-matrix from appropriate n-point function, we need to amputate complete external propagator, replaced by polarization tensor.

17.7 Renormalized perturbation theory

We have been computing in two steps,

- Compute the bare Green's function.

- Rescale to obtain Green's function of renormalized fields. Re-express in terms of renomralized couplings and masses.

Combine these two steps by expressing Lagrangian directly in terms of renormalized quantities,

$$\mathscr{L} = -\frac{1}{4}(F^0)_{\mu\nu}(F^0)^{\mu\nu} + \bar{\psi}^0\left(i\partial\!\!\!/ - e_0 A\!\!\!/^0 - m_0\right)\psi^0\,, \tag{17.91}$$

where the subscript or superscript 0 means the bare quantity. We define

$$
\begin{aligned}
\psi^0 &= \sqrt{Z_2}\psi^R \\
A^0_\mu &= \sqrt{Z_3}A^R_\mu \\
m_0 &= Z_m m_R \\
e_0 &= Z_e e_R \\
Z_1 &= Z_2\sqrt{Z_3}Z_e \\
Z_i = 1+\delta_i\,, &\quad \delta_i = \mathcal{O}(e_R^2) \quad \text{for} \quad i=1,2,3.
\end{aligned}
\tag{17.92}
$$

The Lagrangian with renormalized quantity is

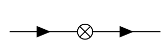
$$
\begin{aligned}
\mathscr{L} &= -\frac{1}{4}Z_3(F^R)_{\mu\nu}(F^R)^{\mu\nu} + Z_2\bar{\psi}^R\left(i\not{\partial} - Z_e\sqrt{Z_3}e_R\not{A}^R - Z_m m_R\right)\psi^R \\
&\overset{Z_i=1+\delta_i}{=} -\frac{1}{4}(F^R)_{\mu\nu}(F^R)^{\mu\nu} + \bar{\psi}^R\left(i\not{\partial} - e_R\not{A}^R - m_R\right)\psi^R - \frac{1}{4}\delta_3(F^R)_{\mu\nu}(F^R)^{\mu\nu} + i\delta_2\bar{\psi}^R\not{\partial}\psi^R \\
&\quad -(\delta_m + \delta_2 + \delta_m\delta_2)\bar{\psi}^R\psi^R - \delta_1 e_R\bar{\psi}_R - \delta_1 e_R\bar{\psi}^R\not{A}^R\psi^R\,.
\end{aligned}
\tag{17.93}
$$

We can absorb $\delta_m\delta_2$ in the redefinition of δ_m to δ'_m such that $\delta'_m + \delta_2 = \delta_m + \delta_2 + \delta_m\delta_2$. In the following discussion δ_m means this new definition.

17.7.1 Feynman rules

Based on the new Lagrangian, we can read the Feynman rules. The same as previous Lagrangian, but with 3 new vertices.

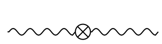
- The first new term is the $\bar{\psi}^R\psi^R$,



$$
= i(\not{p}\delta_2 - (\delta_m + \delta_2)m_R)\,, \tag{17.94}
$$

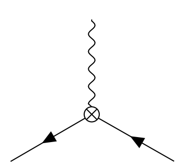
  where we substitute $\not{\partial}$ with $-i\not{p}$ when momentum is aligned with particle flow direction.

- The "free" term of the photon, comes from $\delta_3(F^R_{\mu\nu})(F^R)^{\mu\nu}$,



$$
= -i\delta_3\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\,. \tag{17.95}
$$

- The field-photon interaction comes from the term $\bar{\psi}^R\not{A}^R\psi^R$,



$$
= -i\delta_1 e_R\gamma^\mu\,, \tag{17.96}
$$

  where $\delta_1 e_R \sim e_R^3$.

Remark 17.5. *Perturbation theory is now organized in terms of finite coupling constant e_R, more appealing (though equivalent) to having e_0 appear in intermediate step.*

The δ_i (and the diagrams they generate) are called the counterterms, which is

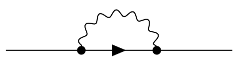
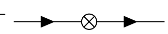
- infinite at each order in e_R^2. (e.g. function of $\frac{1}{\epsilon}$ in dimension regularization).
- coefficient of e_R^{2n} chosen to cancel infinities at given loop order.

17.7.2 Electron self-energy in renormalized perturbation theory

Recall

$$\langle\Omega| \mathscr{T}\left\{\psi(x)\bar{\psi}(y)\right\}|\Omega\rangle = i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} G(\not{p})\,, \tag{17.97}$$

where $\psi, \bar{\psi}$ is the renormalized field and $G(\not{p})$ is formally called G^R. Based on the discussion,

$iG(\not{p}) =$ $+$  $+$  $+\, \mathcal{O}(e_R^4)$

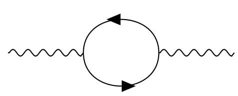
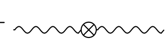
$$= \frac{i}{\not{p}-m_R} + \frac{i}{\not{p}-m_R} i\Sigma_2(\not{p}) \frac{i}{\not{p}-m_R} + \frac{i}{\not{p}-m_R} i\left[\not{p}\delta_2 - (\delta_2+\delta_m)m_R\right]\frac{i}{\not{p}-m_R} + \mathcal{O}(e_R^4)\,, \tag{17.98}$$

where $\Sigma_2(\not{p})$ include e_0 and we use e_R to replace it. We reproduce the previous result for Σ_R.

17.7.3 Photon self-energy (vacuum polarization) in renormalized perturbation theory

The two point function

$$\langle\Omega| \mathscr{T}\left\{A^\mu(x)A^\nu(y)\right\}|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} iG^{\mu\nu}(p)\,. \tag{17.99}$$

$iG^{\mu\nu}(p) =$ $+$  $+$  $+\, \mathcal{O}(e_R^4)\,.$ (17.100)

With

$$iG^{\mu\nu}_{\text{tree}} = -i\frac{g^{\mu\nu}-(1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2-i\epsilon}\,, \tag{17.101}$$

we have

$$\begin{aligned} iG^{\mu\nu} &= iG^{\mu\nu}_{\text{tree}} + iG^{\mu\rho}_{\text{tree}}\left[-i(p^2 g_{\rho\sigma}-p_\rho p_\sigma)e_R^2\Pi_2(p^2)\right] iG^{\sigma\nu}_{\text{tree}} + iG^{\mu\rho}_{\text{tree}}\left[-i\delta_3(p^2 g_{\rho\sigma}-p_\rho p_\sigma)\right] iG^{\sigma\nu}_{\text{tree}} + \mathcal{O}(e_R^4) \\ &= iG^{\mu\nu}_{\text{tree}} + iG^{\mu\rho}_{\text{tree}}\left[-i(p^2 g_{\rho\sigma}-\frac{p_\rho p_\sigma}{p^2})\right] iG^{\sigma\nu}_{\text{tree}} p^2\left(e_R^2\Pi_2(p^2)+\delta_3\right) + \mathcal{O}(e_R^4)\,. \end{aligned} \tag{17.102}$$

The $\Pi_2(p^2)$ contributes to the infinite part $\frac{1}{12\pi^2}\frac{2}{\epsilon}$. The divergence can be absorbed by δ_3.

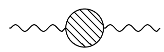
Remark 17.6. *The product of tensors can be simplified if we use Lorenz gauge $\xi = 0$. The key point here is that the nominator of such $G^{\mu\nu}_{tree}$ is a projector,*

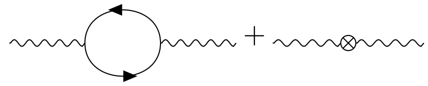
$$\left(g^{\mu\rho}-\frac{p^\mu p^\rho}{p^2}\right)\left(g_\rho{}^\nu - \frac{p_\rho p^\nu}{p^2}\right) = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\,. \tag{17.103}$$

Thus, in Lorenz gauge,

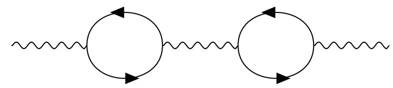
$$iG^{\mu\nu} = i\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2}\left[1+(-i)^2\left(e_R^2\Pi_2(p^2)+\delta_3\right)+\mathcal{O}(e_R^4)\right]\,. \tag{17.104}$$

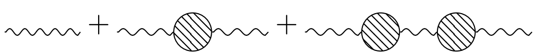
To define the on-shell renormalization scheme, we need to repeat the process as in the electron self-energy part. The 1PI diagram is

$$i\Sigma(\not{p}) =$$  $$= \sum_{\text{1PI diagrams}}$$

$$=$$  $$+ \dots \tag{17.105}$$

$$= -i(p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(p^2)\,,$$

where $\Pi(p^2) = e_R^2 \Pi_2(p^2) + \delta_3 + \dots$ is the form factor. And the prefactor $(p^2 g^{\mu\nu} - p^\mu p^\nu)$ directly comes from the Ward identity since $p_\mu(p^2 g^{\mu\nu} - p^\mu p^\nu) = 0$. Note that $\Pi(p^2)$ does not exhibit a pole at $p^2 = 0$, such poles arise from propagators, carrying the momentum p, e.g.  has a internal photon propagator, gives $\frac{1}{p^2}$, but no such a propagator appears in 1PI. This gives

$$iG^{\mu\nu} =$$  $$+ \dots$$

$$\begin{aligned}
&= -i\frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}}{p^2} + (-i)\frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}}{p^2}(-i)^2\Pi(p^2) + (-i)\frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}}{p^2}[\Pi(p^2)]^2 + \dots \\
&= -i\frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}}{p^2}\frac{1}{1 + \Pi(p^2)}\,.
\end{aligned} \tag{17.106}$$

Because of the Ward identity, the position of the pole of the complete propagator is not shifted compared to the free propagator. It lies at $p^2 = 0$, photon remains massless to all orders in perturbation theory. The renormalization condition to be imposed in a on-shell renormalization scheme is to fix residue of complete propagator to be equal to that of free propagator

$$\Pi(p^2)|_{p^2=0} = 0\,. \tag{17.107}$$

Recall that

$$\Pi_2(p^2) = \frac{1}{2\pi^2}\int_0^1 dx(1-x)\left[\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{m_R^2 - p^2 x(1-x)}\right]\,, \tag{17.108}$$

where we substitute m_0 with m_R since we only consider to the order e_R^2. And

$$\Pi(p^2) = e_R^2 \Pi_2(p^2) + \delta_3 + \dots \tag{17.109}$$

$$\Rightarrow \quad \delta_3 = -e_R^2 \Pi_2(0) = -e_R^2 \frac{1}{12\pi^2}\left(\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{m_R^2}\right)\,, \tag{17.110}$$

which is the result got via the on-shell scheme. Then

$$\Pi(p^2) = \frac{e_R^2}{2\pi^2}\int_0^1 dx x(1-x)\ln\frac{m_R^2}{m_R^2 - p^2 x(1-x)}\,, \tag{17.111}$$

- $\Pi(p^2)$ is finite (to this order).

- In on-shell scheme, $m_R = m_P$, the result is $\tilde{\mu}$ independent.

Renormalizing two point functions has fixed $\delta_2, \delta_m, \delta_3$, the only counterterm is δ_1 from $Z_1 = Z_2\sqrt{Z_3}Z_e$.

17.7.4 Vertex operation

A prior non-vanishing 3 point functions,

$$\langle\Omega|\,\mathscr{T}\left\{\psi(x_1)\bar{\psi}(x_2)A^\mu(x_3)\right\}|\Omega\rangle\ , \tag{17.112}$$

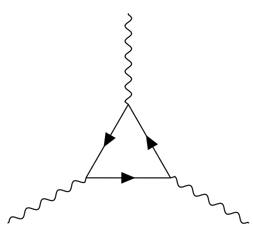
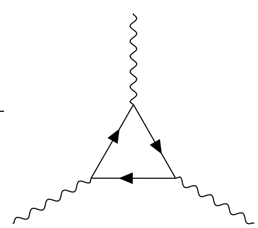
$$\langle\Omega|\,\mathscr{T}\left\{A^\mu(x_1)A^\nu(x_2)A^\rho(x_3)\right\}|\Omega\rangle\ . \tag{17.113}$$

Remark 17.7. *Related to the equation (17.113), by Furry theorem, diagrams with only external photon lines vanish if the number of these lines is odd.*
***Proof**: QED is invariant under $\mathcal{C}$, which is the charge conjugation. By*

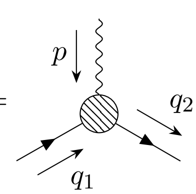
$$\mathcal{C}A^\mu(x)\mathcal{C}^{-1} = -A^\mu(x)\,, \tag{17.114}$$

$$\begin{aligned}\langle\Omega|\,\mathscr{T}\left\{A^{\mu_1}(x_1)\dots A^{\mu_n}(x_n)\right\}|\Omega\rangle &= \langle\Omega|\,\mathcal{C}\mathscr{T}A^{\mu_1}(x_1)\dots A^{\mu_n}(x_n)\mathcal{C}^{-1}|\Omega\rangle \\ &= (-1)^n\langle\Omega|\,\mathscr{T}\left\{A^{\mu_1}(x_1)\dots A^{\mu_n}(x_n)\right\}|\Omega\rangle\ .\end{aligned} \tag{17.115}$$

We can check that using Feynman diagram:


$+$

$= 0\,.$ (17.116)

To address equation (17.112), we define Γ_μ via

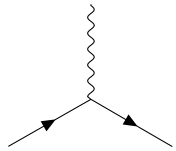
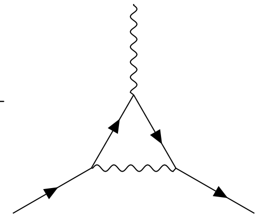
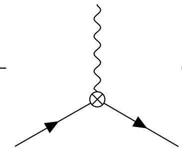
$\bar{u}(q_2)\left[-ie_R\Gamma^\mu(p)\right]u(q_1) =$


, (17.117)

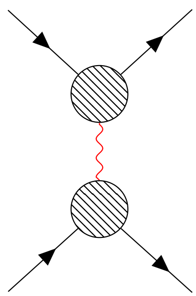
where the blob part means the 1PI diagrams. q_1, q_2 are on-shell and the external photon amputated. Placing q_i on-shell and sandwiching between $\bar{u}(q_2)$ and $u(q_1)$ allows, after some algebra and by involving Ward identity, (and of course, Lorentz invariance), to parametrize the Γ^μ as

$$\Gamma^\mu = F_1\left(\frac{p^2}{m^2}\right)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m}p_\nu F_2\left(\frac{p^2}{m^2}\right)\ , \tag{17.118}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\sigma^\mu, \sigma^\nu]$, F_1 and F_2 are form factors.

$-ie_R\Gamma^\mu(p^2) =$  $+$  $+$  $(-ie_R\delta_1\gamma^\mu) + \ldots .$ (17.119)

δ_1 occurs in appropriate expression to absorb the divergence of F_1. F_2 is finite at this order. The on-shell renormalization scheme requires $\Gamma^\mu(0) = \gamma^\mu$. This implies that e_R coincides with measured electric charge via 2 to 2 scattering. The full diagram,



$$\xrightarrow{p^2\to 0} -e_R^2\bar{u}(q_2)\gamma^\mu u(q_1)\left(-i\frac{g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}}{p^2}\right)\bar{u}(k_2)\gamma^\nu u(k_1)\,, \tag{17.120}$$

where the blob vertex means the 1PI diagram, the red propagator emphasises that it is the complete propagator and $p^2 \to 0$ corresponds to the large distance limit. Performing the calculation (based on the $\Gamma^\mu(0) = \gamma^\mu$), we find $\delta_1 = \delta_2$. This equation holds in any gauge invariant normalization scheme.

Summary of on-shell renormalization conditions

- Impose $m_R = m$ leads to $\Sigma(m_P) = 0$.
- Impose residue of complete fermion propagator equal i leads to $\Sigma'(m_P) = 0$.
- Impose equality of residue of complete propagator and free propagator leads to $\Pi(0) = 0$.
- Impose $e_R = e_P$ be measured by 2-2 scattering leads to $\Gamma^\mu(0) = \gamma^\mu$.

17.7.5 $Z_1 = Z_2$: a consequence of gauge invariance

Recall the Lagrangian

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\psi}\partial\!\!\!/\psi - e_RZ_1\bar{\psi}A\!\!\!/\psi - Z_2Z_mm_R\bar{\psi}\psi \\ &= -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + Z_2\bar{\psi}(i\partial\!\!\!/ - e_R\frac{Z_1}{Z_2}A\!\!\!/)\psi - Z_2Z_mm_R\bar{\psi}\psi\,,\end{aligned} \tag{17.121}$$

where all the quantities are renormalized quantities. For gauge symmetry

$$\begin{aligned}\psi &\to e^{-i\alpha(x)}\psi \\ A^\mu &\to A^\mu + \frac{1}{e_R}\partial^\mu\alpha\,,\end{aligned} \tag{17.122}$$

to survive in the quantum correction, we must have $Z_1 = Z_2$.

Remark 17.8. *The dependence on e_R can be absorbed into the gauge field A^μ,*

$$\begin{aligned} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &\to -\frac{1}{4e_R^2}F_{\mu\nu}F^{\mu\nu} \\ e_R\bar{\psi}\not{A}\psi &\to \bar{\psi}\not{A}\psi \\ \left(A^\mu \to A^\mu + \frac{1}{e_R}\partial^\mu\alpha\right) &\mapsto (A^\mu \to A^\mu + \partial^\mu\alpha) \end{aligned} \tag{17.123}$$

Remark 17.9. *From $Z_1 = Z_2$, we immediately have $\sqrt{Z_3}Z_e = 1$, $\sqrt{Z_3} = \frac{1}{Z_e}$. This is why we could renormalize photon self energy via field δ_3 or via charge δ_e renormalization. These two ways are equivalent.*

In particular, charge renormalization depends only on the photon, and charge ratios that are preserved by quantum corrections. However, gauge invariance is obscured by choice of renormalization scheme. A more formal proof is to prove via Ward-Takahashi identity.

17.8 Renormalization scheme, scalar, and renormalization group equations

What happen to choosing a scale at which we match to experiment?

- In on-shell scheme, m_P and $p^2 \to 0$ are the scales underlying this scheme. The n-point function is independent of μ.
- in MS, $\overline{\text{MS}}$ scheme, n point function in terms of m_R, e_R retain μ-dependence. (or $\tilde{\mu}$), but

$$m_R = m_P + \Sigma_R(m_R) = m_P\left[1 - \frac{\alpha}{4\pi}\left(4 + 3\ln\frac{\tilde{\mu}^2}{m_P^2}\right) + \mathcal{O}(\alpha^2)\right]. \tag{17.124}$$

Observables $\mathscr{O}$ must be μ-independent,

$$\frac{d}{d\mu}\mathscr{O} = 0\,. \tag{17.125}$$

This leads to the renormalization group equations (RGE). Bare quantities must be μ-independent. So the RGE fro charge follows from

$$\begin{aligned} 0 \overset{!}{=} \mu\frac{d}{d\mu}e_0 &= \mu\frac{d}{d\mu}\left(\mu^{\frac{\epsilon}{2}}e_R Z_e\right) \\ \Leftrightarrow \quad \mu\frac{d}{d\mu}e_R &= \beta(e_R)\,, \end{aligned} \tag{17.126}$$

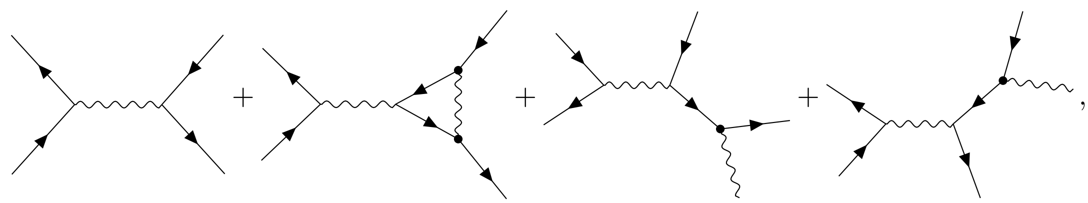
where $\beta(e_R)$ is called the β-function, computed in perturbation theory, in this case $\beta(e_R) = \frac{e_R^3}{12\pi^2}$ to the leading order.

17.9 Infrared divergence

When computing Z_1, Z_2 in the on-shell scheme, the integral of the Feynman parameter $\int \ldots dx$ is divergent. It can be traced to $k^2 \sim 0$ region, which is called the IR divergence. We regulate this divergence by introducing a photon mass m_γ, in violation of gauge invariance. The source of these divergence is that we cannot distinguish between

$$\begin{aligned} e^+e^- &\to e^-e^+ \\ e^-e^+ &\to e^-e^+\gamma\,, \end{aligned} \tag{17.127}$$

for arbitrary soft (low energy) or collinear (in direction of electron) at any finite detector resolution. The similar thing to UV divergence is that it arises in unobservable quantity, e.g. S-matrix with only e^-e^+ in outgoing channel. The difference is that the divergence cancels in cross-section computed for process with different external legs,



(17.128)

The IR divergence is in phase space integral.

Theorem 17.1. *Kinoshita-Lee-Nauenberg theorem: The infrared divergence cancel in unitary theories when soft quanta both in the incoming and outgoing channel are taken to account,*

Remark 17.10. *In QED, it is sufficient to include only soft quanta in outgoing channel. (Bloch–Nordsieck cancellation.)*

17.10 Renormalizability

17.10.1 Superficial degree of divergence

In a 1PI diagram, all internal lines carry momentum that is integrated over. We can estimate superficial degree of divergence D of a diagram defined via

$$\text{diagram} \sim \int^\infty k^{D-1} dk\,, \tag{17.129}$$

- $D > 0$: power-law divergence,
- $D = 0$: logarithm divergence,
- $D < 0$: convergence,

by counting number of interactions vs. number of internal propagators.

Data of diagram required

- Γ_f, the number of internal lines of filed type f.

  Example 17.7. *For QED, f is photon or electron.*

- Mass dimension of propagator $[\langle 0| \mathscr{T} \left\{ \phi_f^*(x)\phi_f(x) \right\} |0\rangle]$.

$$
\begin{aligned}
2[\phi_f] = 2d_f = [\langle 0| \mathscr{T} \left\{ \phi_f^*(x)\phi_f(x) \right\} |0\rangle] = [\int d^4p] + [\Delta_f(p)] = 4 + [\Delta_f(p)] \\
\Rightarrow \quad [\Delta_f(p)] = 2d_f - 4 \,, \quad \Delta_f(p) \sim k^{2d_f - 4} \,.
\end{aligned}
\tag{17.130}
$$

Example 17.8. $\Delta_\phi \sim k^{2\times 1 - 4} = k^{-2}$, $\Delta_\psi = k^{2\times\frac{3}{2}-4} \sim k^{-1}$.

- N_i, the number of vertices of type i, with n_{if} fields of type f.

Example 17.9. *For QED, there is only one type of vertex, with* $n_A = 1, n_\psi = 2$.

- E_f, the number of external lines of type f.

Using these data, we can see that

$$
D = 4 \times [\text{number of integration}] + \sum_f \Gamma_f(-4 + 2d_f) \,. \tag{17.131}
$$

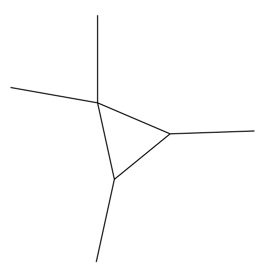
Notice that

$$
[\text{number of integration}] = \sum_f \Gamma_f - (\sum_i N_i - 1) \,, \tag{17.132}
$$

where N_i comes from the momentum conservation δ-function at each vertex and the -1 comes from the overall momentum δ function. We can remove Γ_f dependence in D by involving

$$
2\Gamma_f + E_f = \sum_i N_i n_{if} \,. \tag{17.133}
$$

Example 17.10.



$$
\sim \begin{cases} E_f = 4 \\ \Gamma_f = 3 \\ N_x = 1 \quad n_{xf} = 4 \\ N_y = 2 \quad n_{yf} = 3 \end{cases} \,. \tag{17.134}
$$

Remark 17.11. *In QED,* $2\Gamma_\psi + E_\psi = 2N$*,* E_ψ *must be even.*

This gives

$$
D = 4 - \sum_f d_f E_f - \sum_i N_i \left(4 - \sum_i n_{if} d_f \right) \,. \tag{17.135}
$$

We define $\Delta_i = N_i(4 - \sum_i n_{if} d_f)$, is the mass dimension of coupling of interaction giving via to the vertex

$$
\left[\int d^4x g_i \prod \phi_{if}^{n_{if}} \right] = 0 \,. \tag{17.136}
$$

For $\Delta_i = 0$, only a finite number of n point functions. (i.e. a finite of configuration $\{E_f\}$).

Definition 17.3. *An interaction is*

- *super renormalizable:* $\Delta_i > 0$.
- *renormalizable:* $\Delta_i = 0$.
- *non-renormalizable:* $\Delta_i < 0$

17.10.2 Cancelling divergence

What momentum dependence can a divergence due to a loop integral have? TO answer this question, we first claim that the coefficient of divergence is polynomial in external momenta. We argue that the divergent integral will have external momentum p dependence in denominator, e.g.

$$I(p) = \int_0^\infty \frac{dk}{k+p} \,. \tag{17.137}$$

Differentiating with regard to p lowers the degree of divergence, finally we arrive at a finite integral, e.g.

$$\frac{d}{dp} I(p) = -\int_0^\infty \frac{dk}{(k+p)^2} = -\frac{1}{p} \,. \tag{17.138}$$

Integrating with regard to p yields the original integral, with divergent coefficient multiplying monomials in p, e.g

$$I(p) = -\ln p + c \,, \tag{17.139}$$

where c is the integral constant (divergent). More generally

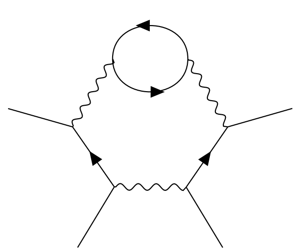
$$\int_0^1 \frac{k^D dk}{k+p} = \sum_{n=0}^{D} a_n p^n + c p^D \ln p \,, \tag{17.140}$$

where the coefficient a_n diverges and c is finite. Monomial p-dependence cab be generated via derivative interactions. To cancel divergence $\propto p^D$ from Green's function with E_f external legs of type f, we introduce an operator with E_f fields of f and k derivative of Lagrangian for $k = 0, \ldots, D$. Dimension of coupling constant of this operator $4 - \sum E_f d_f - k$. Now assume that the interaction generating the divergence was (super) normalizable, which means that

$$D \leq 4 - \sum_f E_f d_f \quad \Rightarrow \quad 4 - \sum_f E_f d_f - k \geq 0 \,, \tag{17.141}$$

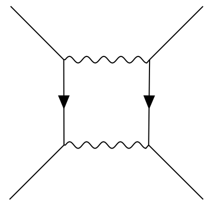
the operators required to absorb all divergence are (super) renormalizable. By introducing all such operators in the Lagrangian, all divergence are cancelled. Why these arguments are not complete?

- There exists sub divergence: superficially convergent diagram may has sub-diagrams that are divergent, e.g.



$$\sim 4 - 4 \times \frac{3}{2} = -2 \,, \tag{17.142}$$

but the diagram is divergent (has a loop), it will cancels with



(17.143)

- Overlapping divergence.

Theorem 17.2. *The BPHZ theorem: A normalizable theory (i.e. a theory containing only renormalizable or super renormalizable interactions), introducing counter terms for all (super) renormalizable interactions is sufficient to absorb all divergence.*

Example 17.11. *In QED, $\Delta_e = 0$, which is a renormalizable theory.*

- *For $\langle\Omega| \mathscr{T}\left\{\psi(x)\bar{\psi}(y)\right\}|\Omega\rangle$, the mass dimension is $4 - 2 \times \frac{3}{2} = 1$, we introduce $\delta_m(k=0), \delta_2(k=1)$.*
- *For $\langle\Omega| \mathscr{T}\left\{A^\mu(x)A^\nu(y)\right\}|\Omega\rangle$, $4 - 2 \times 1 = 2$, we introduce $\delta_3(k=2)$.*
- *For $\langle\Omega| \mathscr{T}\left\{\psi(x)\bar{\psi}(y)A^\mu(z)\right\}|\Omega\rangle$, $4 - 2 \times \frac{3}{2} - 1 = 0$, we introduce $\delta_1(k=0)$.*
- *For $\langle\Omega| \mathscr{T}\left\{A^\mu(x)A^\nu(y)A^\rho(z)A^\sigma(w)\right\}|\Omega\rangle$, $4 - 4 \times 1 = 0$, which is finite.*

Remark 17.12. *There exists non-renormalizable theory, for example, the gravity.*

$$\mathscr{L} = M_{pl}^2 \sqrt{g} R \tag{17.144}$$

with the coupling constant $1/M_{pl}$, M_{pl} is the Planck mass. It is non-renormalizable, which requires an infinite number of counter terms with higher and higher order of derivatives to absorb all divergence. These higher order terms suppressed by $(E/M_{pl})^n$ and there is non-analytical momentum dependence generated by loops.

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