# ADM formalism and D+1-dimensional gravity

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2021

## Contents



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# 1 Introduction

## 1.1 Brief review of gravitational model

General relativity is the foundation of modern theoretical physics. The key equation, which is the well-known *Einstein field equation*, is the starting point of some classical gravity research. The Einstein field equation is

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \,, \tag{1.1}$$

where $g_{\mu\nu}$ is the spacetime metric, the elementary quantity to describe the gravity. $R_{\mu\nu}$ and $R$ is the Ricci tensor and Ricci scalar computed by $g_{\mu\nu}$. $\kappa$ is the coupling constant, usually in the unit of $c = 1$, we set it as $8\pi G$ (in order to match with the Newtonian gravity). $T_{\mu\nu}$ is the energy momentum tensor and $\Lambda$ is the cosmological constant. Throughout this paper, we only consider the universe with $\Lambda = 0$.

In order to prepare the discussion later and for completeness, we briefly review the basic mathematics in general relativity. The gravity emerges from curved spacetime. In Riemannian manifold equipped with metric $g_{\mu\nu}$, we can define the covariant derivative $\nabla$ with respect to a specific connection $\Gamma^{\alpha}_{\beta\gamma}$. The difference between connection coefficient can be eliminated by coordinate transformation so we just choose the convenient convention, which is the Levi-Civita symbol, leading $\nabla_\alpha g_{\beta\gamma} = 0$. The corresponding Christoffel symbol is

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left( \partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\beta\sigma} - \partial_\sigma g_{\beta\gamma} \right) . \tag{1.2}$$

The curvature is described by the non-commutativity of parallel transports, which is the definition of Riemann tensor ${R^i}_{lkm}$ : $(\nabla_k \nabla_m - \nabla_m \nabla_k) A_l = {R^i}_{lkm} A_i$, where $A_i$ is an arbitrary 1-form. Then the Riemann tensor is

$${R^i}_{lkm} = \partial_k \Gamma^i_{lm} - \partial_m \Gamma^i_{kl} + \Gamma^i_{kn}\Gamma^n_{lm} - \Gamma^i_{mn}\Gamma^n_{kl} \,. \tag{1.3}$$

We could then define the corresponding Ricci tensor and Ricci scalar, which is $R_{\mu\rho} = {R^\sigma}_{\mu\sigma\rho}$, $R = g^{\mu\rho}R_{\mu\rho}$. The Einstein field equation comes from the combination of least action principle and Einstein-Hilbert $S_{\mathrm{EH}}$

$$S_{\mathrm{EH}} = \int d^{D+1}x\sqrt{-g}(R - 2\Lambda), \quad \Lambda \equiv 0 \,. \tag{1.4}$$

The variation with respect to $g^{\mu\nu}$ will reproduce the Einstein field equation (1.1). Though it looks simple, the underlying partial differential equations are highly non-linear and complicated.

## 1.2 Questions and the structure of the paper

Nowadays the mainstream physics thinks that our real universe is 4 dimensional. However, the formulation above doesn't impose any restriction of the spacetime dimension. In other words, there might be "gravity" in spacetime of lower or higher dimension. A natural question raises here: What happen if we consider the gravity in 2+1 D spacetime (two spatial dimension and one time dimension, sometimes called three dimensional gravity)? As we can expect, the degrees of freedom in 2+1 D gravity will be much less than that in 4 dimension, which means that it might be exactly solvable. What about the higher dimensional gravity? Is it possible to get analytical solution in higher dimensional gravitational model? The research on these topics has long history and it is impossible to introduce all of them in this paper. We will focus our attention on several simple questions. Those discussion can help us have some intuition toward the gravitational model. These questions are

- What does the 2+1 D dimensional gravity look like? Is there any nontrivial dynamical effect in this dimension?
- Suppose we consider the massive point source model in $D+1$ ($D \geq 2$) dimensional gravity, can we provide the general solution of this model?

Before we discuss these questions before, we should remind that the degrees of freedom of the metric in Einstein gravity theory is not the general $(D+1)(D+2)/2$ (the degrees of freedom of the real symmetric metric). We have the following theorem below

**Theorem 1.1.** *The real degrees of freedom in Einstein gravity theory is*

$$d.o.f = \frac{D(D-1)}{2} - 1\,. \tag{1.5}$$

The proof of this theorem is in the appendix A. For 2+1 dimensional gravity, $D = 2$, there is no dynamical degrees of freedom in the theory (no gravitational wave!). It means that for any gravitational field, the metric can be always deformed to the Minkowski metric. The gravitational behaviour in 2+1 D is so special, in the following discussion, we will explore

- Why the 2+1 D gravity is so special? What's the deep (geometric) reason behind this speciality?

In order to answer these questions, we first introduce ADM formulation of general relativity in $D+1$ dimensional gravity. This is the content of section (2). After constructing ADM formalism in arbitrary dimension, we will couple it with the matter field to rewrite the Einstein field equation in the presence of matter source and matter current. In particular, we assume the matter field is the point source at rest in order to make further progress. These final system of equation is written down in section (3). Then the task is to solve this equation to get the general form of the metric, this is finished in section (4). We prove that the solution is exactly the Schwarzschild solution in $D+1$ dimension. In the section (5), we will discuss the three dimensional gravity, explaining why it is so special. In the last section, we will have a brief summary and discussion about the future work.

## 2 ADM formalism

### 2.1 Brief review of classical mechanics

In classical mechanics, the important quantity is the Lagrangian $L$, which is the function of generic coordinate $q_i$ and its time derivative $L = L(q_i, \dot{q}_i)$. The action is $S = \int L dt$. Based on the least action principle, from $\delta S = 0$, we can get the Euler-Lagrangian equation that dominates the dynamic of the system.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0\,. \tag{2.1}$$

As the usual textbooks suggest, the generic coordinate is the metric $g^{\mu\nu}$, the variation of the action (1.4) with respect to the metric will generate the Einstein field equation (1.1). On the other hand, the equivalent mechanism is the Hamiltonian formalism. Define the conjugate momentum $p_i$ as $p_i = \frac{\partial L}{\partial \dot{q}_i}$, and the Hamiltonian of the system is $H(p_i, q_i) = \sum_i p_i \dot{q}_i - L$. The system of the equations that dominates the dynamic of the system is the well know canonical equation

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,. \tag{2.2}$$

This the basic idea of the ADM formalism, named for its authors Richard Arnowitt, Stanley Deser and Charles W. Misner. We would like to write the dynamical equation of $g^{\mu\nu}$ is this form. The advantage of this formalism is that it can split the time and spatial component, which is easy to deal with and sufficiently use the symmetry to simplify the equations. Also, the Hamiltonian formalism can help us find the way to quantize the gravity, which is the most important problem in modern theoretical physics.

### 2.2 ADM formalism

In usual context the ADM formalism is discussed in 4 dimensional spacetime[1]. However, in arbitrary $D+1$ dimension, there is a little bit difference. We need to extend the formalism to arbitrary dimension. Mostly we just show the key steps and calculation results in the main text. Most of the details are put in the appendix B. Here we use $D$ to denote the spatial dimension and the whole spacetime dimension is $D+1$. It is convenient to decompose the metric as

$$g_{00} = -\alpha^2 + \gamma^{ij}\beta_i\beta_j,\, g_{0i} = g_{i0} = \beta_i,\, g_{ij} = \gamma_{ij}, \tag{2.3}$$

where the Latin indices run from 1 to $D$. $\gamma^{ij}$ is the inverse of $\gamma_{ij}$, i.e. $\gamma^{ik}\gamma_{kj} = \delta^i_j$. This $D+1$ decomposition of the metric replaces the metric components by the lapse function $\alpha(x)$, the shift vector $\beta_i(x)$, and the symmetric spatial metric $\gamma_{ij}(x)$, where we use $x$ to denote the point in the spacetime manifold. The inverse spacetime metric components are

$$g^{00} = -\frac{1}{\alpha^2},\, g^{0i} = \frac{\beta^i}{\alpha^2},\, g^{ij} = \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2}, \tag{2.4}$$

where $\beta^i \equiv \gamma^{ij}\beta_j$. All Latin indices are raised and lowered using the spatial metric. The determinant of the whole spacetime metric is $g = \det(g_{\mu\nu}) = -\alpha^2\gamma$ (The lemma B.1 in the appendix B). The $D+1$ decomposition separates the treatment of time and space coordinates. In the following discussion, we use

$\nabla_i$ to denote the $D$ dimensional covariant derivative with respect to $\gamma$, i.e

$$\nabla_k \gamma_{ij} = 0\,. \tag{2.5}$$

And we use $\gamma^i_{jk}$ to denote its Christoffel symbol, which is

$$\gamma^i_{jk} \equiv \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right)\,. \tag{2.6}$$

In this $D+1$ approach, spacetime is described by a set of $D$-dimensional hypersurfaces of constant time $t = x^0$ propagating forward in time. The intrinsic curvature, described by the Riemann tensor, is

$$\left(\nabla_k\nabla_m - \nabla_m\nabla_k\right)V_l = {}^{(D)}R^i{}_{lkm}V_i\,, \tag{2.7}$$

or the component description,

$${}^{(D)}R^i{}_{lkm} = \partial_k\gamma^i_{lm} - \partial_m\gamma^i_{kl} + \gamma^i_{kn}\gamma^n_{lm} - \gamma^i_{mn}\gamma^n_{kl}\,. \tag{2.8}$$

The $D$ spatial dimensional Ricci tensor is ${}^{(D)}R_{ij} = {}^{(D)}R^k{}_{ikj}$ and the Ricci scalar ${}^{(D)}R = \gamma^{ij}\,{}^{(D)}R_{ij}$. The left subscript $(D)$ emphasises that it is the geometric quantity of the $D$ dimensional hypersurface. In addition to the intrinsic curvature, the hyperspace of constant time also has an extrinsic curvature (or second fundamental form) $K_{ij}$ is

$$K_{ij} = \frac{1}{2\alpha}\left(\nabla_i\beta_j + \nabla_j\beta_i - \partial_t\gamma_{ij}\right)\,. \tag{2.9}$$

The relation between the intrinsic and extrinsic curvature of the constant-time hypersurfaces and the $D+1$ dimensional Riemannian tensor based on Gauss-Codazzi equations

$${}^{(D+1)}R^0{}_{jkl} = -\frac{1}{\alpha}\left(\nabla_k K_{jl} - \nabla_l K_{jk}\right)\,, \tag{2.10}$$

and

$${}^{(D+1)}R^i{}_{jkl} = {}^{(D)}R^i{}_{jkl} - {}^{(D+1)}R^0{}_{jkl}\beta^i + K^i{}_k K_{jl} - K^i{}_l K_{jk}\,. \tag{2.11}$$

The other component of the $D+1$ dimension Riemann tensor is

$${}^{(D+1)}R^0{}_{i0j} = -\frac{1}{\alpha}\partial_t K_{ij} - K_i{}^k K_{jk} - \frac{1}{\alpha}\nabla_i\nabla_j\alpha + \frac{1}{\alpha}\left[\nabla_j\left(\beta^k K_{ik}\right) + K_{jk}\nabla_i\beta^k\right]\,, \tag{2.12}$$

and

$${}^{(D+1)}R^k{}_{i0j} = {}^{(D+1)}R^k{}_{ilj}\beta^l + \left[{}^{(D+1)}R^0{}_{ilj}\beta^l - {}^{(D+1)}R^0{}_{i0j}\right]\beta^k + \alpha\left[\nabla^k K_{ij} - \nabla_i K^k{}_j\right]\,. \tag{2.13}$$

The Einstein tensor defined by $G_{\mu\nu} = {}^{(D+1)}R_{\mu\nu} - \frac{1}{2}{}^{(D+1)}Rg_{\mu\nu}$ is

$$G^{00} = -\frac{\mathscr{H}}{2\alpha^2\sqrt{\gamma}}\,, \quad \mathscr{H} = \sqrt{\gamma}\left[K_{ij}K^{ij} - K^2 - {}^{(D)}R\right]\,, \tag{2.14}$$

$$G^{0i} = \frac{\alpha\mathscr{H}^i + \beta^i\mathscr{H}}{2\alpha^2\sqrt{\gamma}}\,, \mathscr{H}^i = 2\sqrt{\gamma}\nabla_j\left(K^{ij} - K\gamma^{ij}\right)\,, \tag{2.15}$$

$$G^{ij} = -\frac{\beta^i\beta^j\mathscr{H}}{2\alpha^2\sqrt{\gamma}} + \frac{1}{\alpha\sqrt{\gamma}}\partial_t\left(\sqrt{\gamma}P^{ij}\right) + {}^{(D)}R^{ij} - \frac{1}{2}{}^{(D)}R\gamma^{ij} \tag{2.16}$$

$$-\frac{1}{\alpha}\left(\nabla^i\nabla^j - \gamma^{ij}\nabla^2\right)\alpha + \frac{1}{\alpha}\nabla_k\left(\beta^iP^{jk} + \beta^jP^{ik} - \beta^kP^{ij}\right) \tag{2.17}$$

$$+2P^i_{\ k}P^{jk} - PP^{ij} - \frac{1}{2}\left(P^{kl}P_{kl} - \frac{1}{2}P^2\right)\gamma^{ij}\,, \tag{2.18}$$

where $K := \gamma_{ij}K^{ij}$,

$$P^{ij} := K\gamma^{ij} - K^{ij}, \quad P := \gamma_{ij}P^{ij}\,. \tag{2.19}$$

The components of the $D+1$ dimensional Riemann and Einstein tensors are raised and lowered using the $D+1$-dimensional metric. Components of all other quantities like $K_{ij}$, $P_{ij}$, ${}^{(D)}R^i_{\ jkl}$ are raised and lowered using $\gamma_{ij}$. The $D+1$ dimensional Ricci scalar is

$$\sqrt{-g}{}^{(D+1)}R = \alpha\sqrt{\gamma}\left[K_{ij}K^{ij} - K^2 + {}^{(D)}R\right] - 2\partial_t(\sqrt{\gamma}K) + 2\partial_i\left[\sqrt{\gamma}(K\beta^i - \nabla^i\alpha)\right]\,. \tag{2.20}$$

The last two terms are total derivative, if we treat it as the Lagrangian, it will be the surface term, which doesn't contribute to the equations of motion. So the Lagrangian density $\mathscr{L}_{\text{ADM}}$ is

$$\mathscr{L}_{\text{ADM}} = \alpha\sqrt{\gamma}\left[K_{ij}K^{ij} - K^2 + {}^{(D)}R\right]\,, S_{\text{ADM}}[\alpha, \beta_i, \gamma_{ij}] = \int d^{D+1}x\mathscr{L}_{\text{ADM}}\,. \tag{2.21}$$

Here we have already used the units that $16\pi G = 1$[1].

**Remark 2.1.** *One useful observation is that in the ADM Lagrangian there are no dependence on $\dot{\alpha}$ and $\dot{\beta}^i$, which means that these variables are not dynamical variables. The only non-trivial dynamical variable is $\gamma_{ij}$. The canonical momentum conjugated to $\gamma_{ij}$ is $\pi^{ij}$, which is defined as*

$$\pi^{ij} = \frac{\partial\mathscr{L}_{ADM}}{\partial\dot{\gamma}_{ij}}\,. \tag{2.22}$$

*Noticing that the time derivative of the $\gamma_{ij}$ only appears in the $K_{ij}$ (Eq. 2.9),*

$$K_{ij} = \frac{1}{2\alpha}\left(\nabla_i\beta_j + \nabla_j\beta_i - \partial_t\gamma_{ij}\right) \quad \Rightarrow \quad \frac{\partial K_{ab}}{\partial\dot{\gamma}_{ij}} = -\frac{1}{2\alpha}\delta_{ai}\delta_{bj} \tag{2.23}$$

*the conjugated momentum $\pi^{ij}$ is*

$$\begin{aligned}\pi^{ij} = \frac{\partial\mathscr{L}_{ADM}}{\partial\dot{\gamma}_{ij}} &= \sqrt{\gamma}\left(2K^{ab}\frac{\partial K_{ab}}{\partial\dot{\gamma}_{ij}} - 2\gamma^{ab}\gamma^{cd}K_{cd}\frac{\partial K_{ab}}{\partial\dot{\gamma}_{ij}}\right)\\ &= -\sqrt{\gamma}\left(K^{ab}\delta_{ai}\delta_{bj} - K\gamma^{ab}\delta_{ai}\delta_{bj}\right) = \sqrt{\gamma}\left[K\gamma^{ij} - K^{ij}\right] = \sqrt{\gamma}P^{ij}\,.\end{aligned} \tag{2.24}$$

[1]In $D+1$ dimensional gravity, the coupling constant is slightly different, for simplicity we let all of them to be 1.

The Hamiltonian density obtained through the Legendre transformation of $\mathscr{L}_{\text{ADM}}$ is (detail is in B.1),

$$\begin{aligned}\mathscr{H}_{\text{ADM}} &= \pi^{ij}\dot{\gamma}_{ij} - \mathscr{L}_{\text{ADM}} \\ &= \sqrt{\gamma}\left[-\alpha\left({}^{(D)}R + K^2 - K_{ij}K^{ij}\right) + 2\beta^i(\nabla_i K - \nabla_j K^j{}_i)\right] + 2\sqrt{\gamma}\nabla_j\left(K\beta^j - K^j{}_i\beta^i\right),\end{aligned} \tag{2.25}$$

The corresponding Hamiltonian is

$$H_{\text{ADM}} = \int_{\Sigma_t} \mathscr{H}_{\text{ADM}} d^D x\,, \tag{2.26}$$

where the $\Sigma_t$ represents the hypersurface of the constant time $t$. The last term of the equation (2.25) is the divergence term, which doesn't contribute to the integral, so the Hamiltonian density can be written as

$$H_{\text{ADM}} = -\int_{\Sigma_t}\left(\alpha C_0 - 2\beta^i C_i\right)\sqrt{\gamma}d^D x\,, \tag{2.27}$$

where

$$C_0 := {}^{(D)}R + K^2 - K_{ij}K^{ij}\,, \tag{2.28}$$

$$C_i := \nabla_j K^j{}_i - \nabla_i K\,. \tag{2.29}$$

The canonical equation is

$$\pi^\alpha = \frac{\partial \mathscr{L}_{\text{ADM}}}{\partial\dot{\alpha}} \equiv 0, \quad \dot{\pi}^\alpha \equiv 0 = -\frac{\delta H_{\text{ADM}}}{\delta\alpha} = C_0, \tag{2.30}$$

$$\pi^i_\beta = \frac{\partial \mathscr{L}_{\text{ADM}}}{\partial\dot{\beta}_i} \equiv 0, \quad \dot{\pi}^i_\beta = -\frac{\delta H_{\text{ADM}}}{\delta\beta_i} = -2C^i \equiv 0, \tag{2.31}$$

$$\dot{\gamma}_{ij} = \frac{\delta H_{\text{ADM}}}{\delta\pi^{ij}}, \quad \dot{\pi}^{ij} = -\frac{\delta H_{\text{ADM}}}{\delta\gamma_{ij}}\,. \tag{2.32}$$

The above system of equations describes the dynamics of the gravitational system with no matter field. One might be confused that if $C_0 = C_i = 0$, $H_{\text{ADM}} = 0$. It is true if the Hamiltonian is on-shell. The physical interpretation is that the total energy of the system is zero since there is no source or current. However, this doesn't mean that the dynamics is trivial. The dynamical part is included in the last equation. We want the explicit form of these two equations. Recall that the relation between $\pi^{ij}$ and $K^{ij}$ and the important relation $\gamma^{ab}\gamma_{ab} = D$:

$$\begin{aligned}&\pi^{ij} = \sqrt{\gamma}\left[K\gamma^{ij} - K^{ij}\right] \quad \Rightarrow \pi := \gamma_{ij}\pi^{ij} = \sqrt{\gamma}(D-1)K \quad \Rightarrow \quad K = \frac{\pi}{\sqrt{\gamma}(D-1)}\,, \\ &\Rightarrow \quad K^{ij} = \frac{1}{\sqrt{\gamma}}\left(\frac{\pi}{D-1}\gamma^{ij} - \pi^{ij}\right) \quad \Rightarrow \quad K_{ab} = \frac{1}{\sqrt{\gamma}}\left(\frac{\pi}{D-1}\gamma_{ab} - \gamma_{ai}\pi^{ij}\gamma_{jb}\right). \\ &\Rightarrow \quad \frac{\delta K_{ab}}{\delta\pi^{ij}} = \frac{1}{\sqrt{\gamma}}\left(\frac{\gamma_{ab}}{D-1}\frac{\delta\pi}{\delta\pi^{ij}} - \gamma_{ai}\gamma_{jb}\right) = \frac{1}{\sqrt{\gamma}}\left(\frac{\gamma_{ab}\gamma_{ij}}{D-1} - \gamma_{ai}\gamma_{jb}\right)\end{aligned} \tag{2.33}$$

In order to compute $\frac{\delta H}{\delta\pi^{ij}}$, we need to compute $\frac{\delta C_0}{\delta\pi^{ij}}$ and $\frac{\delta C_k}{\delta\pi^{ij}}$ separately. It is easy to see that $\frac{\delta C_k}{\delta\pi^{ij}} \equiv 0$ because $\frac{\delta K_{ab}}{\delta\pi^{ij}} \sim \gamma_{ij}$ and $\nabla_m\gamma_{ij} \equiv 0$. The rest part is (detail is in B.2)

$$\frac{\delta C_0}{\delta\pi^{ij}} = \frac{\delta\,{}^{(D)}R}{\delta\pi^{ij}} + 2K\frac{\delta K}{\delta\pi^{ij}} - 2K^{ab}\frac{\delta K_{ab}}{\delta\pi^{ij}} = \frac{1}{\sqrt{\gamma}}\left(2K_{ij} - \frac{2K\gamma_{ij}}{D-1}\right) \tag{2.34}$$

Then

$$\dot{\gamma}_{ij} = \frac{\delta H}{\delta \pi^{ij}} = -2\alpha \left( K_{ij} - \frac{K\gamma_{ij}}{D-1} \right) . \tag{2.35}$$

Expanding the following equation, we have

$$\dot{\gamma}_{ij} = -(\nabla_i \beta_j + \nabla_j \beta_i) + \dot{\gamma}_{ij} + \frac{2\alpha K \gamma_{ij}}{D-1} \quad \Rightarrow \quad (\nabla_i \beta_j + \nabla_j \beta_i) = \frac{2\alpha K \gamma_{ij}}{D-1} , \tag{2.36}$$

$$\Rightarrow \quad \frac{\delta H}{\delta \pi^{ij}} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \, . \tag{2.37}$$

If we write the equation using the canonical variable, then

$$\dot{\gamma}_{ij} = \nabla_i \beta_j + \nabla_j \beta_i - \frac{2\alpha}{\sqrt{\gamma}} \left( \frac{\pi}{D-1} \gamma_{ij} - \pi_{ij} \right) . \tag{2.38}$$

Now comes to the most complicated equation $\dot{\pi}_{ij} = \frac{\delta H}{\delta \gamma^{ij}}$. In order to compute it, we first rewrite the Hamiltonian with the canonical variables by expanding all the $K_{ab}$ dependence. The terms in the Hamiltonian noe become

$$\begin{aligned}
K^{ab} K_{ab} &= \frac{1}{\gamma} \left( \frac{\pi}{D-1} \gamma^{ab} - \pi^{ab} \right) \left( \frac{\pi}{D-1} \gamma_{ab} - \pi_{ab} \right) = \frac{1}{\gamma} \left( \frac{2-D}{(D-1)^2} \pi^2 + \pi^{ab} \pi_{ab} \right) , \\
K^2 &= \frac{\pi^2}{\gamma (D-1)^2} \quad \Rightarrow \quad K^2 - K^{ab} K_{ab} = \frac{1}{\gamma} \left[ \frac{1}{(D-1)} \pi^2 - \pi^{ab} \pi_{ab} \right] , \\
C_i &= \nabla_j K^j{}_i - \nabla_i K = \nabla_j \left( \frac{\pi}{\sqrt{\gamma}(D-1)} \delta^j_i - \frac{\pi^j{}_i}{\sqrt{\gamma}} \right) - \nabla_i \left( \frac{\pi}{\sqrt{\gamma}(D-1)} \right) = -\nabla_j \left( \frac{\pi^j{}_i}{\sqrt{\gamma}} \right) .
\end{aligned} \tag{2.39}$$

the Hamiltonian is

$$\begin{aligned}
H_{\mathrm{ADM}} &= \int_{\Sigma_t} \left[ -\alpha \sqrt{\gamma} \left( {}^{(D)}R + K^2 - K_{ij} K^{ij} \right) - 2\sqrt{\gamma} \beta^i \nabla_j \left( \frac{\pi^j{}_i}{\sqrt{\gamma}} \right) \right] d^D x \\
&= -\int_{\Sigma_t} \alpha \sqrt{\gamma}\, {}^{(D)}R d^D x - \int_{\Sigma_t} \alpha \frac{1}{\sqrt{\gamma}} \left[ \frac{1}{(D-1)} \pi^2 - \pi^{ab} \pi_{ab} \right] d^D x - \int_{\Sigma_t} 2\sqrt{\gamma} \beta^i \nabla_j \left( \frac{\pi^j{}_i}{\sqrt{\gamma}} \right) d^D x
\end{aligned} \tag{2.40}$$

We want to compute the variation with respect to $\gamma_{ij}$, some building blocks are needed,

$$\begin{aligned}
&\delta \gamma = \gamma \gamma^{ij} \delta \gamma_{ij} \, , \quad \delta \sqrt{\gamma} = \frac{1}{2\sqrt{\gamma}} \delta \gamma = \frac{1}{2} \sqrt{\gamma} \gamma^{ij} \delta \gamma_{ij} \, , \quad \gamma_{ij} \delta \gamma^{ij} = -\gamma^{ij} \delta \gamma_{ij} \, , \\
&\delta \gamma^{ij} = -\gamma^{ia} \delta \gamma_{ab} \gamma^{bj} \, , \quad \delta \left( \frac{1}{\sqrt{\gamma}} \right) = -\frac{1}{2} \gamma^{-3/2} \delta \gamma = -\frac{1}{2\sqrt{\gamma}} \gamma^{ij} \delta \gamma_{ij} \, ,
\end{aligned} \tag{2.41}$$

### 2.2.1 The variation of the first term

The variation of the first term in the Hamiltonian is standard, which just reproduces the Einstein field equation within the $D$ dimensional hypersurface.

$$\begin{aligned}
\delta \left( \sqrt{\gamma}\, {}^{(D)}R \right) &= \delta \left( \sqrt{\gamma} \gamma^{ij\,(D)} R_{ij} \right) = \left( \delta \sqrt{\gamma} \right) {}^{(D)}R + \sqrt{\gamma} \delta \gamma^{ij\,(D)} R_{ij} + \sqrt{\gamma} \gamma^{ij} \delta^{(D)} R_{ij} \, , \\
&= \sqrt{\gamma} \left( -{}^{(D)}R^{ij} + \frac{1}{2} {}^{(D)}R \gamma^{ij} \right) \delta \gamma_{ij} + \sqrt{\gamma} \gamma^{ij} \delta^{(D)} R_{ij} \, .
\end{aligned} \tag{2.42}$$

The last term needs special attention. After complicated simplification (detail is in B.3), combining with the spatial integral it will finally give

$$-\int_{\Sigma_t} \alpha\sqrt{\gamma}\gamma^{ij}\delta^{(D)}R_{ij}d^Dx = \int_{\Sigma_t} \sqrt{\gamma}\left[(\nabla^k\nabla_k\alpha)\gamma^{ij} - \nabla^j\nabla^i\alpha\right]\delta\gamma_{ij}d^Dx\,. \tag{2.43}$$

The variation of the first term in the Hamiltonian (2.40) is finished.

### 2.2.2 The variation of the second term

Now comes to the second term. Before we start, we need to prepare some equations first. It is obvious that the second term contains conjugated momentum dependence. Similar to the case in classical mechanics, the conjugated momentum $\pi^{ij}$ should be viewed as the independent variable, which means that $\delta\pi^{ij} \equiv 0$. since we use $\gamma_{ij}$ as the basic variable (generic coordinate), But $\delta\pi_{ij}$ is not zero. In this convention we calculate the variable

$$\delta\pi^2 = 2\pi\delta\pi = 2\pi\delta\left(\gamma_{ij}\pi^{ij}\right) = 2\pi\pi^{ij}\delta\gamma_{ij}\,,\quad \delta\pi^{ab}\pi_{ab} = \delta\left(\pi^{ab}\gamma_{ai}\pi^{ij}\gamma_{jb}\right) = 2\pi^j{}_k\pi^{ki}\delta\gamma_{ij}\,. \tag{2.44}$$

The variation with respect to the second term becomes

$$\begin{aligned}
&-\delta\int_{\Sigma_t}\alpha\frac{1}{\sqrt{\gamma}}\left[\frac{1}{(D-1)}\pi^2 - \pi^{ab}\pi_{ab}\right]d^Dx \\
&= -\int_{\Sigma_t}\alpha\delta\left(\frac{1}{\sqrt{\gamma}}\right)\left[\frac{1}{(D-1)}\pi^2 - \pi^{ab}\pi_{ab}\right]d^Dx - \int_{\Sigma_t}\frac{\alpha}{\sqrt{\gamma}}\left[\frac{1}{(D-1)}2\pi\pi^{ij}\delta\gamma_{ij} - 2\pi^j{}_k\pi^{ki}\delta\gamma_{ij}\right]d^Dx \\
&= +\int_{\Sigma_t}\frac{\alpha}{2\sqrt{\gamma}}\gamma^{ij}\left[\frac{1}{D-1}\pi^2 - \pi^{ab}\pi_{ab}\right]\delta\gamma_{ij}d^Dx - \int\frac{2\alpha}{\sqrt{\gamma}}\left(\frac{\pi\pi^{ij}}{D-1} - \pi^j{}_k\pi^{ki}\right)\delta\gamma_{ij}d^Dx
\end{aligned} \tag{2.45}$$

In fact, this is the only different term compared to the usual $3+1$ ADM formalism.

### 2.2.3 The variation of the third term

Now we left with the third term in the Hamiltonian (2.40). Before we start to vary it with respect to metric, we should do some simplification,

$$\begin{aligned}
-\int_{\Sigma_t} 2\sqrt{\gamma}\beta^i\nabla_j\left(\frac{\pi^j{}_i}{\sqrt{\gamma}}\right)d^Dx &= -\int_{\Sigma_t} 2\sqrt{\gamma}\left[\nabla_j\left(\frac{\beta^i\pi^j{}_i}{\sqrt{\gamma}}\right) - \frac{\pi^j{}_i}{\sqrt{\gamma}}\nabla_j\beta^i\right]d^Dx\,, \\
&= \text{surface term} + \int_{\Sigma_t} 2\pi^j{}_i\nabla_j\beta^i d^Dx = \text{surface term} + \int_{\Sigma_t} 2\pi^{ij}\nabla_i\beta_j d^Dx
\end{aligned} \tag{2.46}$$

Again, the surface term can be dropped. We focus on the variation of the non trivial term. There is a new variable in this expression ($\beta^i$). We have freedom to choose which one ($\beta_i$ or $\beta^i$) is the generic coordinate, as long as we keep the calculation consistent. We choose $\beta^i$ as the generic coordinate for consistence. Now we have

$$\pi^{ij}\delta(\nabla_i\beta_j) = \pi^{ij}\nabla_i\beta^k\delta\gamma_{jk} + \pi^{ij}\gamma_{jk}\beta^l\delta\Gamma^k_{il}\,. \tag{2.47}$$

The second term is

$$\pi^{ij}\gamma_{jk}\beta^l\frac{1}{2}\gamma^{km}\left(\nabla_i\delta\gamma_{ml}+\nabla_l\delta\gamma_{im}-\nabla_m\delta\gamma_{il}\right)=\pi^{ij}\beta^l\frac{1}{2}\left(\nabla_i\delta\gamma_{jl}+\nabla_l\delta\gamma_{ij}-\nabla_j\delta\gamma_{il}\right)=\pi^{ij}\beta^l\frac{1}{2}\nabla_l\delta\gamma_{ij}\,. \tag{2.48}$$

Putting the whole term in the integral (detail is in B.4)

$$\delta\int_{\Sigma_t}2\pi^{ij}\nabla_i\beta_j d^Dx=\int_{\Sigma_t}2\pi^{k(j}\nabla_k\beta^{i)}\delta\gamma_{ij}d^Dx-\int_{\Sigma_t}\nabla_l\left(\pi^{ij}\beta^l\right)\delta\gamma_{ij}d^Dx\,, \tag{2.49}$$

Now we have already overcome all the difficulties and are ready to provide the explicit form of the variation

$$\begin{aligned}\delta H=\int_{\Sigma_t}&\left[\alpha\sqrt{\gamma}\left({}^{(D)}R^{ij}-\frac{1}{2}{}^{(D)}R\gamma^{ij}\right)+\sqrt{\gamma}\left[(\nabla^k\nabla_k\alpha)\gamma^{ij}-\nabla^j\nabla^i\alpha\right]+\right.\\&\left.\frac{\alpha}{2\sqrt{\gamma}}\gamma^{ij}\left(\frac{1}{D-1}\pi^2-\pi^{ab}\pi_{ab}\right)-\frac{2\alpha}{\sqrt{\gamma}}\left(\frac{\pi\pi^{ij}}{D-1}-\pi^j{}_k\pi^{ki}\right)+2\pi^{k(j}\nabla_k\beta^{i)}-\nabla_l\left(\pi^{ij}\beta^l\right)\right]\delta\gamma_{ij}d^Dx\,.\end{aligned} \tag{2.50}$$

The canonical equation for $\dot{\pi}^{ij}$ is

$$\begin{aligned}\dot{\pi}^{ij}=-\frac{\delta H}{\delta\gamma_{ij}}=-&\left[\alpha\sqrt{\gamma}\left({}^{(D)}R^{ij}-\frac{1}{2}{}^{(D)}R\gamma^{ij}\right)+\sqrt{\gamma}\left[(\nabla^k\nabla_k\alpha)\gamma^{ij}-\nabla^j\nabla^i\alpha\right]+\right.\\&\left.\frac{\alpha}{2\sqrt{\gamma}}\gamma^{ij}\left(\frac{1}{D-1}\pi^2-\pi^{ab}\pi_{ab}\right)-\frac{2\alpha}{\sqrt{\gamma}}\left(\frac{\pi\pi^{ij}}{D-1}-\pi^j{}_k\pi^{ki}\right)+2\pi^{k(j}\nabla_k\beta^{i)}-\nabla_l\left(\pi^{ij}\beta^l\right)\right]\,.\end{aligned} \tag{2.51}$$

# 3 Couple to matter field

## 3.1 General argument

Above we have built the ADM formalism in vacuum. According to the additive property of the Lagrangian and action, if we want to couple the theory with matter field, the full Lagrangian of the gravitational system is

$$\mathscr{L}=\mathscr{L}_{\mathrm{ADM}}+\mathscr{L}_M\,, \tag{3.1}$$

where the $\mathscr{L}_M$ is the Lagrangian contributed by the matter field and the matter field also depends on the metric field $\alpha,\beta^i,\gamma_{ij}$. The full action becomes

$$S=S_{\mathrm{ADM}}+S_M=\int\mathscr{L}_{\mathrm{ADM}}d^{D+1}x+\int\mathscr{L}_M d^{D+1}x\,. \tag{3.2}$$

Based on the stationary-action principle, the variation with respect to the metric $g_{\mu\nu}$ should be zero, i.e. $\delta S=0$. The variation with respect to the second term will gives us the energy momentum tensor. The $D+1$ dimensional energy momentum tensor is

$$T^{\mu\nu}:=\frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}}\,,\quad\delta\mathscr{L}_M=\frac{\sqrt{-g}}{2}T^{\mu\nu}\delta g_{\mu\nu}\,. \tag{3.3}$$

## 3.2 Particle source

In order to make further progress, we choose a specific Lagrangian in the following calculation. Our purpose is to find the point particle solution in arbitrary $D+1$ dimension, so the action of the matter $S_M$ should be [4],

$$S_M = -\int M d\tau \sqrt{-g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu}\,, \tag{3.4}$$

where introduce extra generic coordinate $q^\mu$ and fix the gauge that $q^0 = \tau(\dot{q}^0 \equiv 1)$. The Lagrangian (not the Lagrangian density) of the matter and the corresponding conjugate momentum $\pi_i^q$ (keep in mind that the subscript $q$ emphasises that this the quantity relates to the particle) is

$$L_M = -M\sqrt{-g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu} \quad \Rightarrow \quad \pi_i^q = \frac{\partial L_M}{\partial \dot{q}^i} = -M\frac{1}{2}\frac{1}{\sqrt{-g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu}}(-2g_{i\nu}\dot{q}^\nu) = \frac{Mg_{i\nu}\dot{q}^\nu}{\sqrt{-g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu}}\,. \tag{3.5}$$

Notice that $\dot{q}^0 \equiv 1$ and the metric could be written as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)\,, \tag{3.6}$$

we could rewrite the conjugate momentum $\pi_i^q$ and the Lagrangian (detail is in B.5)

$$\begin{aligned} \pi_i^q &= M\frac{\gamma_{ij}\left(\dot{q}^j + \beta^j\right)}{\sqrt{\alpha^2 - \gamma_{ij}(\dot{q}^i + \beta^i)(\dot{q}^j + \beta^j)}} \\ L_M &= (\dot{q}^i + \beta^i)\pi_i^q - \alpha\sqrt{\gamma^{ij}\pi_i^q\pi_j^q + M^2} \end{aligned} \tag{3.7}$$

We need the Lagrangian density $\mathscr{L}_M$, which we could obtain by adding the delta function,

$$\begin{aligned} \int \mathscr{L}_M d^{D+1}x = \int d\tau L_M &= \int d\tau d^{D+1}x \left[(\dot{q}^i + \beta^i)\pi_i^q - \alpha\sqrt{\gamma^{ij}\pi_i^q\pi_j^q + M^2}\right]\delta^{(D+1)}(x^\mu - q^\mu) \\ &= \int d^{D+1}x\left[(\dot{q}^i + \beta^i)\pi_i^q - \alpha\sqrt{\gamma^{ij}\pi_i^q\pi_j^q + M^2}\right]\delta^{(D)}(\vec{x} - \vec{q})\,, \end{aligned} \tag{3.8}$$

while the last equal holds for we integrate over the $\tau$. As a result, the Lagrangian density is

$$\mathscr{L}_M = \left[(\dot{q}^i + \beta^i)\pi_i^q - \alpha\sqrt{\gamma^{ij}\pi_i^q\pi_j^q + M^2}\right]\delta^{(D)}(\vec{x} - \vec{q})\,. \tag{3.9}$$

For preparing the calculation below, we compute the variation with respect to $\alpha, \beta_i$ and $\delta\gamma_{ij}$ .

$$\delta_\alpha \mathscr{L}_M = -\sqrt{\gamma^{ij}\pi_i^q\pi_j^q + M^2}\,\delta^{(D)}(\vec{x} - \vec{q})\delta\alpha \tag{3.10}$$

$$\delta_{\beta_i}\mathscr{L}_M = \pi_i^q\delta(\beta^i)\delta^{(D)}(\vec{x} - \vec{q}) = \pi^{qi}\delta^{(D)}(\vec{x} - \vec{q})\delta\beta_i \tag{3.11}$$

$$\delta_{\gamma_{ij}}\mathscr{L}_M = -\frac{\alpha\pi^{qi}\pi^{qj}\delta^{(D)}(\vec{x} - \vec{q})\delta\gamma_{ij}}{2\sqrt{\gamma_{ij}\pi^{iq}\pi^{qj} + M^2}}\,. \tag{3.12}$$

Now we want to compute the variation about $\mathscr{L}_{\text{ADM}}$. Since we have already compute the variation about $\mathscr{H}_{\text{ADM}}$, we just need to apply the Legendre transformation that $\mathscr{L} = \sum \pi\dot{\gamma} - \mathscr{H}$ to get the results. Pay attention that $\alpha$ and $\beta_i$ is not dynamical variable $\dot{\alpha} = \dot{\beta}^i \equiv 0$. The ADM action is

$$S_{\text{ADM}} = \int dt\left(\int \pi^{ij}\dot{\gamma}_{ij}d^D x - H_{\text{ADM}}\right)\,, \tag{3.13}$$

variation with respect to the whole generic coordinate becomes

$$\delta S_{\mathrm{ADM}} = \int dt \left[ \int_{\Sigma_t} \left( \pi^{ij} \delta\dot{\gamma}_{ij} + \dot{\gamma}_{ij} \delta\pi^{ij} \right) d^D x - \delta H_{\mathrm{ADM}} \right] . \tag{3.14}$$

In the last section we have already finished the calculation of $\delta H_{\mathrm{ADM}}$. Based on the relation $\delta S_{\mathrm{ADM}} + \delta S_M = 0$, we could combine all these stuff and write down the complete dynamical equations of the system. Variation with respect to $\alpha$ and $-\delta_\alpha \mathscr{H}_{\mathrm{ADM}} + \delta_\alpha \mathscr{L}_M = 0$ give

$$\sqrt{\gamma}\,{}^{(D)}R + \frac{1}{\sqrt{\gamma}} \left[ \frac{1}{D-1}\pi^2 - \pi^{ab}\pi_{ab} \right] - \sqrt{\gamma^{ij}\pi^q_i \pi^q_j + M^2}\,\delta^{(D)}(\vec{x} - \vec{q}) = 0 . \tag{3.15}$$

Variation with respect to $\beta_i$ and $-\delta_{\beta_i} \mathscr{H}_{\mathrm{ADM}} + \delta_{\beta_i} \mathscr{L}_M = 0$ give

$$2\sqrt{\gamma}\nabla_j \left( \frac{\pi^{ji}}{\sqrt{\gamma}} \right) + \pi^{qi} \delta^{(D)}(\vec{x} - \vec{q}) = 0 . \tag{3.16}$$

Variation with respect to $\delta\gamma_{ij}$ and $-\dot{\pi}^{ij}\delta\gamma_{ij} - \delta_{\gamma_{ij}} \mathscr{H}_{\mathrm{ADM}} + \delta_{\gamma_{ij}} \mathscr{L}_M$ give

$$\begin{aligned} &-\Bigg[ \alpha\sqrt{\gamma} \left( {}^{(D)}R^{ij} - \frac{1}{2}{}^{(D)}R\gamma^{ij} \right) + \sqrt{\gamma} \left[ (\nabla^k \nabla_k \alpha)\gamma^{ij} - \nabla^j \nabla^i \alpha \right] + \frac{\alpha}{2\sqrt{\gamma}}\gamma^{ij} \left( \frac{1}{D-1}\pi^2 - \pi^{ab}\pi_{ab} \right) \\ &\quad - \frac{2\alpha}{\sqrt{\gamma}} \left( \frac{\pi\pi^{ij}}{D-1} - \pi^j{}_k \pi^{ki} \right) + 2\pi^{k(j}\nabla_k \beta^{i)} - \nabla_l \left( \pi^{ij}\beta^l \right) \Bigg] - \dot{\pi}^{ij} - \frac{\alpha\pi^{qi}\pi^{qj}\delta^{(D)}(\vec{x} - \vec{q})}{2\sqrt{\gamma_{ij}\pi^{iq}\pi^{qj} + M^2}} = 0 . \end{aligned} \tag{3.17}$$

Variation with respect to $\pi^{ij}$ and $\dot{\gamma}_{ij} - \delta_{\pi^{ij}} \mathscr{H}_{\mathrm{ADM}} = 0$ give

$$\dot{\gamma}_{ij} = \nabla_i \beta_j + \nabla_j \beta_i + \frac{2\alpha}{\sqrt{\gamma}} \left( \pi_{ij} - \frac{\pi}{D-1}\gamma_{ij} \right) . \tag{3.18}$$

Variation with respect to $q^i$ and $\pi^q_i$ needs a little bit work. Since it is not related to the gravitational term, we can focus on the variation of $S_M$. The Hamiltonian density is

$$\begin{aligned} \mathscr{H}_M &= \pi^q_i \dot{q}^i \delta^{(D)}(\vec{x} - \vec{q}) - \left[ (\dot{q}^i + \beta^i)\pi^q_i - \alpha\sqrt{\gamma^{ij}\pi^q_i \pi^q_j + M^2} \right] \delta^{(D)}(\vec{x} - \vec{q}) \\ &= \left[ -\beta^i \pi^q_i + \alpha\sqrt{\gamma^{ij}\pi^q_i \pi^q_j + M^2} \right] \delta^{(D)}(\vec{x} - \vec{q}) . \end{aligned} \tag{3.19}$$

The Hamiltonian canonical equation is

$$\dot{q}^i = \frac{\delta H_M}{\delta \pi^q_i} = -\beta^i + \left. \frac{\alpha\pi^{qi}}{\sqrt{\gamma^{ij}\pi^q_i \pi^q_j + M^2}} \right|_{\vec{x}=\vec{q}} , \tag{3.20}$$

$$\begin{aligned} \dot{\pi}^q_i &= -\frac{\delta H_M}{\delta q^i} = -\int \left[ -\beta^i \pi^q_i + \alpha\sqrt{\gamma^{ij}\pi^q_i \pi^q_j + M^2} \right] \frac{1}{\delta q^i} \delta^{(D)}(\vec{x} - \vec{q}) d^D x \\ &= \int \delta^{(D)}(\vec{x} - \vec{q}) \frac{\delta}{\delta q^i} \left[ -\beta^i \pi^q_i + \alpha\sqrt{\gamma^{ij}\pi^q_i \pi^q_j + M^2} \right] d^D x \\ &= \partial_i \left[ -\beta^i \pi^q_i + \alpha\sqrt{\gamma^{ij}\pi^q_i \pi^q_j + M^2} \right] \Big|_{\vec{x}=\vec{q}} . \end{aligned} \tag{3.21}$$

The equation (3.15, 3.16, 3.17, 3.18, 3.20,3.21) gives the whole dynamical equations of the $D+1$ dimensional point source gravitational model. To solve this system of equation we need some physical assumption and symmetries to reduce the degrees of freedom.

Before we continue, we would like to emphasise that why we use these equations that looks much more complicated to substitute elegant Einstein field equation (1.1). The reason is that we cannot write down a general form of the Einstein tensor $G_{\mu\nu}$ without assuming the dimension. Besides, our manifold is Lorentz manifold, the signature $(-,+,\ldots,+)$ forbids us to use the conformal symmetry to simplify the equation. In ADM formalism, the time component and spatial components split and we could will take this advantage to calculation the Ricci tensor within the $D$ dimensional hypersurface, which is the basic idea that why we choose ADM formalism.

# 4 Solution of massive point source gravitational model in $D+1$ -dimension

Consider the particle at rest, located at the origin of the coordinate system $q^0 = x^0 = \tau\,, q^i = 0$, $i = 1,\ldots,D$. This system is static and spatially spherically symmetric, which means that $\beta^i \equiv 0$. Eq. (3.20) reduces to $\pi^{qi} \equiv 0$. Eq. (3.21) reduces to

$$0 = \dot{\pi}^q_i = \partial_i \left[ -\beta^i \pi^q_i + \alpha \sqrt{\gamma^{ij}\pi^q_i\pi^q_j + M^2} \right]\Big|_{\vec{x}=\vec{q}} \quad \Rightarrow \quad \partial_i \alpha|_{\vec{x}=0} = 0\,. \tag{4.1}$$

Eq. (3.18) reduces to (static metric)

$$\begin{aligned} 0 = \dot{\gamma}_{ij} = \nabla_i\beta_j + \nabla_j\beta_i + \frac{2\alpha}{\sqrt{\gamma}}\left(\pi_{ij} - \frac{\pi}{D-1}\gamma_{ij}\right) \Rightarrow \pi_{ij} - \frac{\pi}{D-1}\gamma_{ij} = 0 \\ \Rightarrow \quad \pi - \frac{\pi}{D-1}D = 0 \quad \Rightarrow \quad \pi = 0 \quad \Rightarrow \quad \pi_{ij} = 0,. \end{aligned} \tag{4.2}$$

Eq. (3.15) reduces to

$$\sqrt{\gamma}^{(D)}R - M\delta^{(D)}(\vec{x}) = 0\,. \tag{4.3}$$

Eq. (3.16) reduces to the trivial identity. Eq (3.17) reduces to

$$\alpha\sqrt{\gamma}\left({}^{(D)}R^{ij} - \frac{1}{2}{}^{(D)}R\gamma^{ij}\right) + \sqrt{\gamma}\left[(\nabla^k\nabla_k\alpha)\gamma^{ij} - \nabla^j\nabla^i\alpha\right] = 0\,. \tag{4.4}$$

In summary, the non-trivial equations are

$$\sqrt{\gamma}^{(D)}R - M\delta^{(D)}(\vec{x}) = 0\,, \tag{4.5}$$

$$\alpha\sqrt{\gamma}\left({}^{(D)}R^{ij} - \frac{1}{2}{}^{(D)}R\gamma^{ij}\right) + \sqrt{\gamma}\left[(\nabla^k\nabla_k\alpha)\gamma^{ij} - \nabla^j\nabla^i\alpha\right] = 0\,, \tag{4.6}$$

$$\partial_i\alpha|_{\vec{x}=0} = 0\,. \tag{4.7}$$

In the following discussion, we assume that $D \geq 3$ and we left $D = 2$ case in the next section, which is the 2+1 D dimensional gravity, to explain clearly that why it is so special and consider its geometry.

## 4.1 Key equation

Using the spherically symmetric assumption, we set (for simplicity we use the assumption similar to [5])

$$\gamma_{ij} = f^{-2}\left(\sqrt{\delta_{ab}x^a x^b}\right)\delta_{ij} \quad \Rightarrow \quad \sqrt{\gamma} = f^{-D}\,. \tag{4.8}$$

Now the question becomes that what is the general form of Ricci scalar ${}^{(D)}R$ under the conformally flat metric. The general form of the Ricci scalar [5] is

$${}^{(D)}R = (D-1)\left[2f\Delta f - D(\partial f)^2\right] \tag{4.9}$$

where $\Delta$ is the Laplace operator in Euclidean space $\Delta := \sum_{i=1}^{D}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}$, $(\partial f)^2 := \sum_{i=1}^{D}(\partial_i f)^2$. The key equation (4.5) now becomes ($D \geq 2$)

$$\frac{D-1}{f^D}\left[2f\Delta f - D(\partial f)^2\right] = M\delta^{(D)}(\vec{x})\,. \tag{4.10}$$

Now the problem is just solving the PDE. Pay attention that we couldn't multiply $f^D$ to the right hand side because the metric is not well defined at the origin. Thanks to the spherical symmetry $f(x^i) = f(r(x^i))$, we have

$$(\partial f)^2 = \sum_{i=1}^{D}\left(\frac{\partial f}{\partial x^i}\right)^2 = \sum_{i=1}^{D}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x^i}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2\sum_{i=1}^{D}\left(\frac{x^i}{r}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2. \tag{4.11}$$

$$\Delta f = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^{D-1}}f\,, \tag{4.12}$$

where $\Delta_{S^{D-1}}$ is the Laplace-Beltrami operator on ($D$-1)-sphere. Since $f$ has no angular dependence, $\Delta_{S^{N-1}}f \equiv 0$. The equation (4.10) reduces to

$$\frac{D-1}{f^D}\left[2f\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial f}{\partial r}\right) - D\left(\frac{\partial f}{\partial r}\right)^2\right] = M\delta^{(D)}(\vec{x}) \tag{4.13}$$

In order to solve this differential equation, we need to introduce some backgrounds about the Laplace operator, which we provide in the appendix (C). We want to find the general solution for the key equation (4.13). Noticing that the Laplace operator acting on $\frac{1}{r^{D-2}}$ will generates the delta function, which is proportional to the right hand side, we guess the solution takes the following form

$$f(r) = \left(1 + \frac{r_0^{D-2}}{r^{D-2}}\right)^{-\frac{2}{D-2}}\,, \tag{4.14}$$

inspired by the four dimensional Schwartzchild solution). $r_0$ is the integral constant, whose physical interpretation is the horizon radius related to the mass of the black hole. In appendix D , we prove that it is indeed the general solution of the key equation. Here we have already seen that the solution is not valid when $D = 2$. The three dimensional gravity starts to show its speciality.

### 4.2 The solution of the time component

The answer is not complete since we do not solve the $\alpha$. The equation for $\alpha$ is

$$\begin{aligned}
&\alpha\sqrt{\gamma}\left({}^{(D)}R^{ij} - \frac{1}{2}{}^{(D)}R\gamma^{ij}\right) + \sqrt{\gamma}\left[(\nabla^k\nabla_k\alpha)\gamma^{ij} - \nabla^j\nabla^i\alpha\right] = 0\,, \\
&\partial_i\alpha|_{\vec{x}=0} = 0\,.
\end{aligned} \tag{4.15}$$

Contract $\gamma_{ij}$ on the both sides of the first equation, we have

$$\alpha\sqrt{\gamma}\left(1 - \frac{D}{2}\right){}^{(D)}R + \sqrt{\gamma}(D-1)\Delta_\gamma\alpha = 0\,. \tag{4.16}$$

Following the similar process as above, we can get the solution

$$\alpha = \frac{\left(1 - \frac{r_0^{D-2}}{r^{D-2}}\right)}{\left(1 + \frac{r_0^{D-2}}{r^{D-2}}\right)}\,. \tag{4.17}$$

We solve the $D+1$ dimensional point source model analytically. In fact, the result is the same as the $D+1$ dimensional Schwarzschild black hole [3].

## 5 2+1 D gravity

### 5.1 Solution of the 2+1 D gravity

As we obtain in the last section, the general solution of the $D+1$ dimensional gravity is not valid when $D = 2$. The reason is that the differential operator is completely changed due to the reduction of dimension. Consider $f = e^\eta$. $\partial_i f = f\partial_i\eta$, $\partial_i^2 f = f(\partial_i\eta)^2 + f\partial_i^2\eta$, substitute them into the key equation we will have

$$\frac{D-1}{f^D}\left[2f\sum_{i=1}^{D}\left(f(\partial_i\eta)^2 + f\partial_i^2\eta\right) - D\sum_{i=1}^{D}(f\partial_i\eta)^2\right] = \frac{D-1}{f^D}\left[(2-D)f^2\sum_{i=1}^{D}(\partial_i\eta)^2 + 2f^2\sum_{i=1}^{D}\partial_i^2\eta\right]\,. \tag{5.1}$$

It is easy to see that if $D = 2$, the equation is very simple, which is the Poisson equation in the plane

$$2\Delta\eta = M\delta^{(2)}(\vec{x})\,, \tag{5.2}$$

The answer to the question that why $2+1$ dimensional gravity is so special is that the first derivative part disappears after applying the conformal transformation, which totally changes the solution of the differential equations. The solution is easy,

$$\eta = \frac{M}{4\pi}\ln r\,. \tag{5.3}$$

The equation (4.16) now becomes $\Delta\alpha = 0$. Combining with the boundary condition $\partial_i\alpha|_{\vec{x}=0} = 0$, we conclude that $\alpha$ must be constant. By rescaling of the time coordinate, we can set $\alpha = 1$. The full metric is

$$ds^2 = -dt^2 + r^{-\frac{M}{2\pi}}\left(dr^2 + r^2d\theta^2\right) \quad\Rightarrow\quad ds^2 = -dt^2 + r^{-8GM}\left(dr^2 + r^2d\theta^2\right)\,. \tag{5.4}$$

There exists a naked singularity at the origin.

### 5.2 The geometry of one particle solution

The metric is obviously conformally flat. To see it more explicitly, we use another chart

$$\rho = \alpha^{-1} r^{\alpha}, \quad \theta' = \alpha\theta, \quad \alpha \equiv 1 - 4GM \tag{5.5}$$

we could see that $d\rho = r^{\alpha-1}dr$, $d\theta' = \alpha d\theta$, then the metric is

$$ds^2 = -dt^2 + \left(d\rho^2 + \rho^2 d\theta'^2\right) \tag{5.6}$$

which is indeed the flat metric. One thing special here is the range of $\theta'$, which is from 0 to $2\pi\alpha = 2\pi(1-4GM)$. When the mass is not so large $m < \frac{1}{4G}$, the range is well-defined. However, when $M = \frac{1}{4G}$ ($\alpha = 0$), the transformation is singular. And the line element is

$$dl^2 = \frac{1}{r^2}\left(dr^2 + r^2 d\theta^2\right) = (d\ln r)^2 + d\theta^2 \tag{5.7}$$

where $r$ ranges from $(0, +\infty)$ and $\theta \in [0, 2\pi]$, The geometry is a periodic strip or infinite cylinder in "Cartesian" coordinate $(\ln r, \theta)$. For the case $m > 1/4G$ ($\alpha < 0$), the metric is

$$dl^2 = r^{2\alpha-2}dr^2 + r^{2\alpha}d\theta^2 \tag{5.8}$$

If we adopt the transformation $r = \frac{1}{u}$, which maps the center to the infinity and the infinity to the center. The metric is

$$dl^2 = u^{-2\alpha-2}du^2 + u^{-2\alpha}d\theta^2 \tag{5.9}$$

Remember $\alpha < 0$, If we again change the coordinate

$$\rho' = -\alpha^{-1}u^{-\alpha}, \quad \theta'' = -\alpha\theta \tag{5.10}$$

The metric

$$dl^2 = d\rho'^2 + \rho'^2 d\theta''^2 \tag{5.11}$$

is again flat. It is easy to see that there exists a correspondence: when $\alpha < 0$, it is equivalent to the case that we put a particle at infinity with mass $\frac{1}{2}G - M$. If we represent the spacetime using geometrical approach, the deficit angle of the conical spacetime is $\beta = 2\pi - 2\pi\alpha = 8\pi GM$, as the Figure 1 shown. This geometric description tells us it seems that there is no interacting force between two static massive particles since they doesn't curve the spacetime. We will see it clearer in the later discussion.

### 5.3 $N$ point particles solution

It is well known that we cannot get the analytic solution of three-body problem in the four dimensional spacetime. However, due to the locally flatness in 2+1 D gravity, it is easy to find the solution with $N$ massive particles [2]. We don't need to repeat the whole procedure to find solution. A smarter way is to

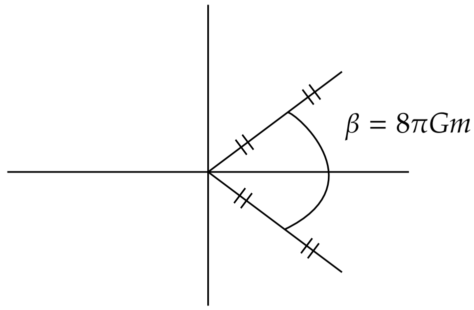



Figure 1: Deficit angle

assume that the metric takes the form

$$g = -dt^2 + e^{2\eta(x,y)}(dx^2 + dy^2)\,. \tag{5.12}$$

The Einstein tensor

$$G^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}G_{\alpha\beta} = \delta^\mu_0\delta^\nu_0 G_{00} = -\delta^\mu_0\delta^\nu_0 e^{-2\eta(x,y)}\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right) = -\delta^\mu_0\delta^\nu_0 e^{-2\eta}\Delta\eta\,. \tag{5.13}$$

To apply the Einstein field equation, we need the energy momentum tensor. Using the argument in section (3), the energy momentum tensor is (the detail is in B.6)

$$T^{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}_M}{\delta g_{\mu\nu}} \quad \Rightarrow \quad T^{00} = \frac{m}{\sqrt{-g}}\delta^{(2)}(x^i) = \frac{m}{\sqrt{-g}}\delta^{(2)}(\vec{x})\,. \tag{5.14}$$

**Remark 5.1.** *A question raises here. Is this assumption correct? It seems that we impose the condition that all the particles will stay at the original position. In fact, it is true consistent. Here we provide a simple argument, which is from [2], This setting for energy momentum tensor obviously satisfies the conservation law* $\nabla_\mu T^{\mu\nu} = 0$*. This is consistent with the Euler-Lagrangian equation, which is the geodesic equation* $\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$*. For spatial part,*

$$\ddot{x}^i + \Gamma^i{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \tag{5.15}$$

*Recall the setting* $\dot{x}^\mu(0) = (1, \vec{0})$*, around* $t = 0$*, the equation reduces to*

$$\ddot{x}^i + \Gamma^i{}_{00} = 0 \quad \Rightarrow \quad \ddot{x}^i = -\frac{1}{2}g^{ij}\partial_j g_{00} \tag{5.16}$$

*We already proved that* $g_{00}$ *is a constant, which means that the spatial accelerations of the particle is constant. So the setting is consistent. But if we set that the initial velocity is non-zero, the geodesic is not straight line. So the interacting force depends on the velocity of the particle. It's very different and strange behaviour in 2+1 D gravity.*

So the non trivial Einstein field equation for the $N$ particle case is

$$G^{00} = -e^{-2\eta}\Delta\eta = \frac{\kappa}{\sqrt{-g}}\sum_n m_n \delta^{(n-1)}(\vec{r}-\vec{r}_n)\,, \tag{5.17}$$

where $\kappa$ is the coupling constant. Noticing $\sqrt{-g} = e^{2\eta}$, the factor before the both sides of the equation can be canceled. And then solution for $\eta$ is just

$$\ln\Phi = -\frac{\kappa}{\pi}\sum_n m_n \ln|\vec{r}-\vec{r}_n| \quad \Rightarrow \quad \Phi = \prod_n |\vec{r}-\vec{r}_n|^{-\frac{\kappa m_n}{\pi}}\,, \tag{5.18}$$

where $m_i$ is the mass of the $i$-th particle. As we show above, the metric is still conformally flat.

### 5.4 The trajectory of the probe particle

To see the different behaviour of the 2+1 D gravity, we calculate the geodesic of the metric (5.4). In order to avoid the singularity of the metric ($r = 0$), we go back to the Cartesian coordinate,

$$ds^2 = -dt^2 + (x^2+y^2)^{-4Gm}(dx^2+dy^2)\,. \tag{5.19}$$

The geodesic equations are

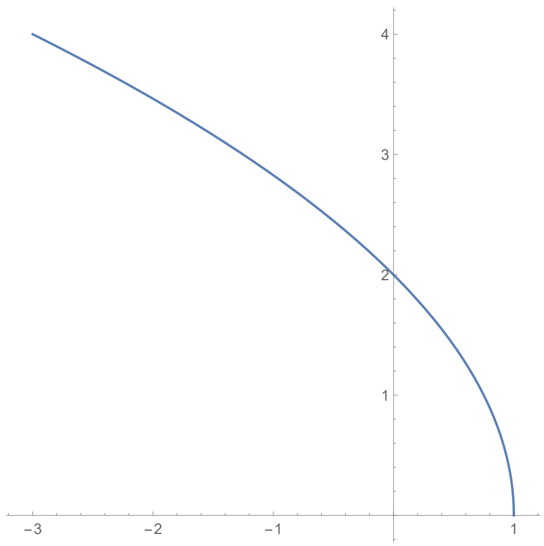
$$\begin{aligned} &\frac{d^2x}{d\tau^2} + \frac{4Gm}{x^2+y^2}\left[-2y\frac{dx}{d\tau}\frac{dy}{d\tau} + x\left(-\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2\right)\right] = 0\,, \\ &\frac{d^2y}{d\tau^2} + \frac{8Gm}{x^2+y^2}\left[-2x\frac{dx}{d\tau}\frac{dy}{d\tau} + y\left(\left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2\right)\right] = 0\,. \end{aligned} \tag{5.20}$$

The numerical solution is as the figure 2 shows.

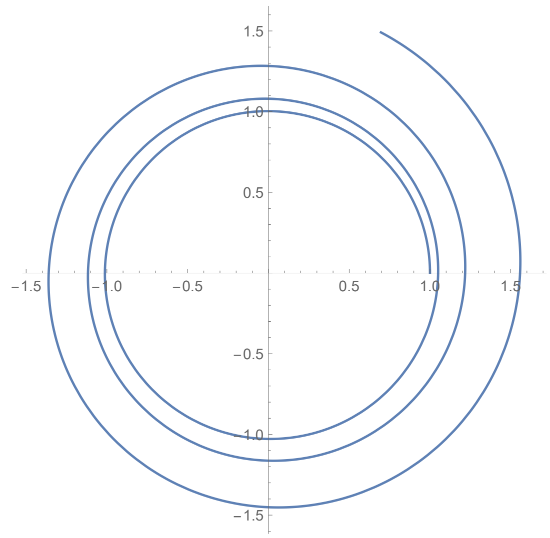
## 6 Summary and discussion

In this paper, we first review the basic knowledge of general relativity and point out that it is not easy to use in the usual field equation formalism. We want to use the spherical symmetry to find the solution of point source model in $D+1$ dimension. Because of the spherical symmetry, the metric only has two degrees of freedom, $\alpha$ and $f$. Notice that the split of the time component and the spatial component can help us investigate the geometry by studying the geometric properties restricting on the hypersurface $\Sigma_t$ of constant time $t$. Such a hypersurface is conformal to the Euclidean space, which helps us find the general formula of the Ricci scalar. This inspires us to develop a new formalism of general relativity by splitting the time and space and use Hamiltonian to describe the system. We emphasises that though ADM formalism is well-known, usually this formalism is set to be in the 3+1 spacetime. We are the first one to write down everything in detail under $D+1$ dimension, carefully fixing the coefficients. ADM formalism is useful for it is similar to what we do in classical mechanics. It might help us in the searching of the way to quantize gravity.

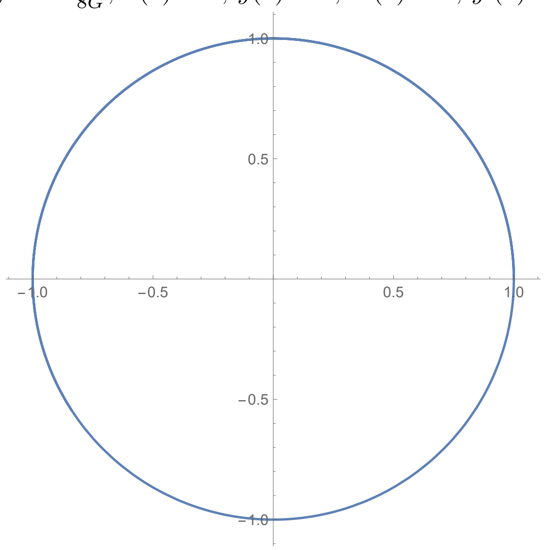
After constructing the $D+1$ dimensional ADM formalism, we couple this formalism with the matter field. Also, we assume the source is just the static massive particle to make further progress. We finally get



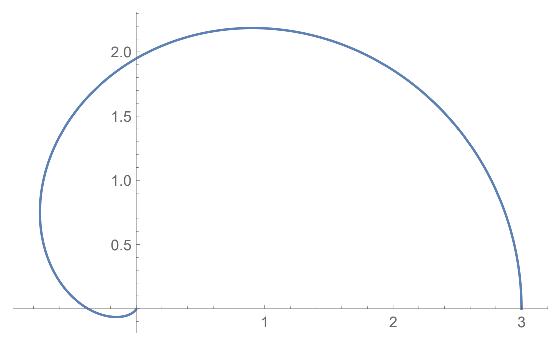

(a) $m = \frac{1}{8G}$, $x(0) = 1$, $y(0) = 0$, $x'(0) = 0$, $y'(0) = 2$




(b) $m = \frac{1}{4.01G}$, $x(0) = 1$, $y(0) = 0$, $x'(0) = 0$, $y'(0) = 2$




(c) $m = \frac{1}{4G}$, $x(0) = 1$, $y(0) = 0$, $x'(0) = 0$, $y'(0) = 2$




(d) $m = \frac{1}{3G}$, $x(0) = 3$, $y(0) = 0$, $x'(0) = 0$, $y'(0) = 5$

Figure 2: Plots of the numerical solution of the ODEs (5.20)

the general solution of the differential equations, consistent with the Schwarzschild solutions in arbitrary dimension. This provide a new track to obtain the Schwarzschild solution.

2+1 D gravity is special and we find the root reason: the form of the differential equation (or the Ricci scalar) changes greatly. We show that the gravitational effect highly relates to the topological property of the spacetime manifold again. In the section 5, we discuss its speciality by computing $N$ particle solution and geodesics. In modern theoretical physics, an important topic is the AdS/CFT correspondence and one important example is $\mathrm{AdS}_3$ corresponds to the $\mathrm{CFT}_2$. This study of three dimensional gravity will give us intuition towards the physics in the research.

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# A Linearized gravity and dynamical degrees of freedom of in Einstein gravity

One important thing is that in three dimensional model, there is no physical degrees of freedom of the metric. One way to prove this claim is to see the linearized approximation, which will dynamically trivial. The linearized approximation is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Here we go through the details of this linearization process for completeness. The Einstein field equation for arbitrary dimension is

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab} \tag{A.1}$$

Variate both two sides of the equation about the metric, we have

$$\delta R_{ab} - \frac{1}{2}(\delta R)g_{ab} - \frac{1}{2}R\delta g_{ab} = \kappa\delta T_{ab} \tag{A.2}$$

Consider it terms by terms, first

$${R_{abc}}^d(\lambda) - {R_{abc}}^d = 2\nabla_{[a}{C^d}_{b]c}(\lambda) - 2{C^b}_{[a|e|}(\lambda){C^e}_{b]c}(\lambda) \tag{A.3}$$

where

$$C^c{}_{ab}(\lambda) = -\frac{1}{2}g^{cd}(\lambda)\left[\nabla_a g_{bd}(\lambda) + \nabla_b g_{da}(\lambda) - \nabla_d g_{ab}(\lambda)\right] \tag{A.4}$$

Using the notation

$$\delta g_{ab} = \left.\frac{dg_{ab}(\lambda)}{d\lambda}\right|_{\lambda=0} \tag{A.5}$$

Then

$$\delta R_{ac} = \left.\frac{dR_{ac}}{d\lambda}\right|_{\lambda=0} = 2\nabla_{[a}\delta C^b{}_{b]c} \tag{A.6}$$

$$\delta C^c{}_{ab} = \left.\frac{dC^c{}_{ab}(\lambda)}{d\lambda}\right|_{\lambda=0} = -\frac{1}{2}\left[\nabla_a \delta g_{bd} + \nabla_b \delta g_{da} - \nabla \delta g_{ab}\right] \tag{A.7}$$

Because $\nabla_a g_{bc}(0) = 0$,

$$\delta R_{ac} = \frac{1}{2}g^{bd}\left[\nabla_b\nabla_a\delta g_{cd} + \nabla_b\nabla_c\delta g_{ad} - \nabla_b\nabla_d\delta g_{ac} - \nabla_a\nabla_c\delta g_{bd}\right] \tag{A.8}$$

$$\delta R = \delta\left(g^{ab}R_{ab}\right) = (\delta g^{ab})R_{ab} + g^{ab}\delta R_{ab} \tag{A.9}$$

$$\delta g^{ab} = -g^{ac}g^{bd}\delta g_{bd} \tag{A.10}$$

we have

$$\begin{aligned} 2\delta G_{ab} = &-\Box h_{ab} + R_{ac}h_b{}^c + R_{bc}h_a{}^c - 2R_{acbd}h^{cd} + \Box h g_{ab} + \nabla_a\nabla_c h_b{}^c \\ &+ \nabla_b\nabla_c h_a{}^c - \nabla_a\nabla_b h - \nabla^c\nabla^d h_{cd}g_{ab} + R^{cd}h_{cd}g_{ab} - Rh_{ab} \end{aligned} \tag{A.11}$$

where $h_{ab} = \delta g_{ab}$, $h = g^{ab}h_{ab}$. If we denote $\bar{h}_{ab} = h_{ab} - \frac{1}{2}hg_{ab}$, the equation is

$$2\delta G_{ab} = -\Box\bar{h}_{ab} + R_{ac}\bar{h}_b{}^c + R_{bc}\bar{h}_a{}^c - 2R_{abcd}\bar{h}^{cd} + \nabla_a\nabla_c\bar{h}_b{}^c + \nabla_b\nabla_c\bar{h}_a{}^c - \nabla^c\nabla^d\bar{h}_{cd}g_{ab} + R^{cd}\bar{h}_{cd}g_{ab} - R\bar{h}_{ab} \tag{A.12}$$

If we see the energy momentum tensor is the perturbation (The background is Minkowski space)

$$-\Box\bar{h}_{ab} + R_{ac}\bar{h}_b{}^c + R_{bc}\bar{h}_a{}^c - 2R_{abcd}\bar{h}^{cd} + \nabla_a\nabla_c\bar{h}_b{}^c + \nabla_b\nabla_c\bar{h}_a{}^c - \nabla^c\nabla^d\bar{h}_{cd}g_{ab} + R^{cd}\bar{h}_{cd}g_{ab} - R\bar{h}_{ab} = 16\pi T_{ab} \tag{A.13}$$

There are some gauge degrees of freedom in the perturbation, we could choose the so called Lorenz gauge:

$$\nabla^a\bar{h}_{ab} = 0 \tag{A.14}$$

Then the equation simplifies to

$$-\Box\bar{h}_{ab} + R_{ac}\bar{h}_b{}^c + R_{bc}\bar{h}_a{}^c - 2R_{abcd}\bar{h}^{cd} + R^{cd}\bar{h}_{cd}g_{ab} - R\bar{h}_{ab} = 16\pi T_{ab} \tag{A.15}$$

Now the background spacetime is Minkowski spacetime, $R_{abcd} \equiv= 0$, $\nabla_a = \partial_a$. The equation reduces to

$$-\Box\bar{h}_{ab} = 16\pi T_{ab} \tag{A.16}$$

$$\partial^a\bar{h}_{ab} = 0 \tag{A.17}$$

In general a symmetric real matrix has $n(n+1)/2$ degrees of freedom. The Lorentz gauge has $n$ equations, which kills $n$ degrees of freedom. However, we still have residual gauge freedom compatible with the Lorenz gauge. Consider the transformation,

$$\bar{h}_{ab} \to \bar{h}_{ab} + \nabla_a \xi_b + \nabla_b \xi_a - (\nabla_c \xi^c)\, g_{ab} \tag{A.18}$$

where $\xi^a$ satisfies

$$\nabla^a \left[\nabla_a \xi_b + \nabla_b \xi_a - (\nabla_c \xi^c) g_{ab}\right] = \nabla^a \nabla_a \xi_b + \nabla_a \nabla_b \xi^a - \nabla_b \nabla_a \xi^a = \nabla^a \nabla_a \xi_b + R_{bc} \xi^c = 0 \tag{A.19}$$

It is still compatible with Lorenz gauge. Then we would have extra $n$ restriction. So the true degrees of freedom is $\frac{n^2-3n}{2}$. Clearly if the $n = 3$, there is no degrees of freedom, which means that the metric is dynamically trivial. If we want non trivial gravity dominated by Einstein field equation, we should consider $n \leq 4$, the magic 4 is the lower bound of the dimension.

# B Calculation supplementary

In this appendix we provide more details of the calculation, supplementing the argument in the paper.

## B.1 ADM formalism

**Lemma B.1.** *The determinant of the whole spacetime metric satisfy $g = -\alpha^2 \gamma$, where $\gamma$ is the determinant of $\gamma_{ij}$.*

**Proof**: It is convenient to consider the determinant of the inverse metric $\det(g) = \frac{1}{\det(g^{-1})}$, the determinant of the inverse metric is

$$\begin{vmatrix} -\frac{1}{\alpha^2} & \frac{\vec{\beta}^T}{\alpha^2} \\ \frac{\vec{\beta}}{\alpha^2} & \gamma^{-1} - \frac{\vec{\beta}\vec{\beta}^T}{\alpha^2} \end{vmatrix} \tag{B.1}$$

where the $\vec{\beta}$ represents the vector $(\beta^1, \beta^2, \ldots, \beta^n)^T$. For i-th row, we times $\beta^i$ of the 0-th row and add to it, which doesn't change the determinant, and it reduces to

$$\begin{vmatrix} -\frac{1}{\alpha^2} & \frac{\vec{\beta}^T}{\alpha^2} \\ \vec{0} & \gamma^{-1} \end{vmatrix} = -\frac{|\gamma^{-1}|}{\alpha^2}\,, \tag{B.2}$$

which proves that $g = -\alpha^2 \gamma$.

**Calculation Detail B.1.** *The Hamiltonian density is*

$$\begin{aligned}
\mathscr{H}_{ADM} &= \pi^{ij}\dot{\gamma}_{ij} - \mathscr{L}_{ADM} \\
&= \sqrt{\gamma}\left[K\gamma^{ij} - K^{ij}\right](-2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i) - \alpha\sqrt{\gamma}\left[K_{ij}K^{ij} - K^2 + {}^{(D)}R\right] \\
&= \sqrt{\gamma}\left[-\alpha\left({}^{(D)}R + K^2 - K_{ij}K^{ij}\right) + 2\left(K\gamma^j{}_i - K^j{}_i \nabla_j \beta^i\right)\right] \\
&= \sqrt{\gamma}\left[-\alpha\left({}^{(D)}R + K^2 - K_{ij}K^{ij}\right) + 2\beta^i(\nabla_i K - \nabla_j K^j{}_i)\right] + 2\sqrt{\gamma}\nabla_j\left(K\beta^j - K^j{}_i\beta^i\right)\,,
\end{aligned} \tag{B.3}$$

**Calculation Detail B.2.** *The variation with respect to $\pi^{ij}$ of $C_0$ is*

$$\frac{\delta C_0}{\delta \pi^{ij}} = \cancel{\frac{\delta^{(D)}R}{\delta \pi^{ij}}} + 2K\frac{\delta K}{\delta \pi^{ij}} - 2K^{ab}\frac{\delta K_{ab}}{\delta \pi^{ij}}\,, \tag{B.4}$$

*since there is no dependence of* $\pi^{ij}$ *is* ${}^{(D)}R$. *Recall that* $\frac{\delta K}{\delta \pi^{ij}} = \gamma^{ab}\frac{\delta K_{ab}}{\delta \pi^{ij}}$, *we have*

$$
\begin{aligned}
\Rightarrow \quad \frac{\delta C_0}{\delta \pi^{ij}} &= \left(2K\gamma^{ab} - 2K^{ab}\right)\frac{1}{\sqrt{\gamma}}\left(\frac{\gamma_{ab}\gamma_{ij}}{D-1} - \gamma_{ai}\gamma_{jb}\right) \\
&= \frac{1}{\sqrt{\gamma}}\left(\cancel{\frac{2K\gamma^{ab}\gamma_{ab}\gamma_{ij}}{D-1}} - 2K\gamma_{ij} - \frac{2K\gamma_{ij}}{D-1} + 2K_{ij}\right) \\
&= \frac{1}{\sqrt{\gamma}}\left(2K_{ij} - \frac{2K\gamma_{ij}}{D-1}\right) .
\end{aligned}
\tag{B.5}
$$

**Calculation Detail B.3.** *Here we show all the detail of the variation*

$$
\gamma^{ij}\delta^{(D)}R_{ij} = \nabla_c\left(\gamma^{ab}\delta\Gamma^c_{ab} - \gamma^{ac}\delta\Gamma^b_{ab}\right) . \tag{B.6}
$$

*In order to compute it, we need the variation of the Christoffel symbol. We could evaluate this value in the following way. For arbitrary point* $x_0$, *in its Riemannian normal coordinate, the Christoffel symbol vanishes at this specific point, its variation is just*

$$
\delta\Gamma^c_{ab} = \frac{1}{2}\eta^{cd}\left(\partial_a\delta g_{db} + \partial_b\delta g_{ad} - \partial_d\delta g_{ab}\right) , \tag{B.7}
$$

*where* $\eta^{ad}$ *is the inverse of Minkowski metric. Recovering the tensorial notation, we have*

$$
\delta\Gamma^c_{ab} = \frac{1}{2}g^{cd}\left(\nabla_a\delta g_{db} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ab}\right) . \tag{B.8}
$$

*Now using our notation, we have*

$$
\delta\Gamma^c_{ab} = \frac{1}{2}\gamma^{cd}\left(\nabla_a\delta\gamma_{db} + \nabla_b\delta\gamma_{ad} - \nabla_d\delta\gamma_{ab}\right) . \tag{B.9}
$$

*The variation with respect to* $\gamma_{ij}$ *of the last term in the equation (2.42)*

$$
\begin{aligned}
&- \int_{\Sigma_t}\alpha\sqrt{\gamma}\gamma^{ij}\delta^{(D)}R_{ij}d^Dx = - \int_{\Sigma_t}\alpha\sqrt{\gamma}\nabla_c\left(\gamma^{ab}\delta\Gamma^c_{ab} - \gamma^{ac}\delta\Gamma^b_{ab}\right)d^Dx \\
&= - \int_{\Sigma_t}\nabla_c\left[\alpha\left(\gamma^{ab}\delta\Gamma^c_{ab} - \gamma^{ac}\delta\Gamma^b_{ab}\right)\right]\sqrt{\gamma}d^Dx + \int_{\Sigma_t}\nabla_c\alpha\left(\gamma^{ab}\delta\Gamma^c_{ab} - \gamma^{ac}\delta\Gamma^b_{ab}\right)\sqrt{\gamma}d^Dx \\
&= -\textit{surface term} + \int_{\Sigma_t}\nabla_c\alpha\left(\gamma^{ab}\delta\Gamma^c_{ab} - \gamma^{ac}\delta\Gamma^b_{ab}\right)\sqrt{\gamma}d^Dx .
\end{aligned}
\tag{B.10}
$$

*At the boundary surface, the variation* $\delta\gamma_{ij}$ *vanishes, which means that we could drop the surface term. This trick will be applied several times in the following calculations. The factor in the integral is*

$$
\begin{aligned}
\gamma^{ab}\delta\Gamma^c_{ab} - \gamma^{ac}\delta\Gamma^b_{ab} &= \gamma^{ab}\frac{1}{2}\gamma^{cd}\left(\nabla_a\delta\gamma_{db} + \nabla_b\delta\gamma_{ad} - \nabla_d\delta\gamma_{ab}\right) - \gamma^{ac}\frac{1}{2}\gamma^{bd}\left(\nabla_a\delta\gamma_{db} + \nabla_b\delta\gamma_{ad} - \nabla_d\delta\gamma_{ab}\right) \\
&= \frac{1}{2}\left(\gamma^{ab}\gamma^{cd} - \gamma^{ac}\gamma^{bd}\right)\left(\nabla_a\delta\gamma_{db} + \nabla_b\delta\gamma_{ad} - \nabla_d\delta\gamma_{ab}\right) ,
\end{aligned}
\tag{B.11}
$$

*Combining it with the other factor, we have*

$$\begin{aligned}
\nabla_c\alpha\left(\gamma^{ab}\delta\Gamma^c_{ab}-\gamma^{ac}\delta\Gamma^b_{ab}\right) &= \frac{1}{2}\nabla_c\alpha\left(\gamma^{ab}\gamma^{cd}-\gamma^{ac}\gamma^{bd}\right)\left(\nabla_a\delta\gamma_{db}+\nabla_b\delta\gamma_{ad}-\nabla_d\delta\gamma_{ab}\right)\\
&= \frac{1}{2}\left(\gamma^{ab}\nabla^d\alpha-\nabla^a\alpha\gamma^{bd}\right)\left(\nabla_a\delta\gamma_{db}+\nabla_b\delta\gamma_{ad}-\nabla_d\delta\gamma_{ab}\right)\\
&= \frac{1}{2}\Big[\nabla^d\alpha\nabla^b\delta\gamma_{db}+\nabla^d\alpha\nabla^a\delta\gamma_{ad}-\nabla^d\alpha\nabla_d(\gamma^{ab}\delta\gamma_{ab})\\
&\quad -\nabla^a\alpha\nabla_a(\gamma^{bd}\delta\gamma_{db})-\nabla^a\alpha\nabla^d\delta\gamma_{ad}+\nabla^a\alpha\nabla^b\delta\gamma_{ab}\Big]\\
&= \nabla^a\alpha\nabla^b\delta\gamma_{ab}-\nabla^a\alpha\nabla_a\left(\gamma^{cd}\delta\gamma_{cd}\right)\\
&= \nabla^b\left(\nabla^a\alpha\delta\gamma_{ab}\right)-(\nabla^b\nabla^a\alpha)\delta\gamma_{ab}-\nabla_a\left(\nabla^a\alpha\gamma^{cd}\delta\gamma_{cd}\right)+\nabla_a\nabla^a\alpha(\gamma^{cd}\delta\gamma_{cd})\,.
\end{aligned} \tag{B.12}$$

*Again, the total derivative term could be ignored, simplified result is*

$$\nabla_c\alpha\left(\gamma^{ab}\delta\Gamma^c_{ab}-\gamma^{ac}\delta\Gamma^b_{ab}\right)=\left[(\nabla^k\nabla_k\alpha)\gamma^{ij}-\nabla^j\nabla^i\alpha\right]\delta\gamma_{ij}\,, \tag{B.13}$$

*while the expression (B.10) becomes*

$$\begin{aligned}
-\int_{\Sigma_t}\alpha\sqrt{\gamma}\gamma^{ij}\delta^{(D)}R_{ij}d^Dx &= \int_{\Sigma_t}\nabla_c\alpha\left(\gamma^{ab}\delta\Gamma^c_{ab}-\gamma^{ac}\delta\Gamma^b_{ab}\right)\sqrt{\gamma}d^Dx\\
&= \int_{\Sigma_t}\sqrt{\gamma}\left[(\nabla^k\nabla_k\alpha)\gamma^{ij}-\nabla^j\nabla^i\alpha\right]\delta\gamma_{ij}d^Dx\,.
\end{aligned} \tag{B.14}$$

*We then finish the proof of the result (2.43).*

**Calculation Detail B.4.** *The details of the expression (2.49) is*

$$\begin{aligned}
\delta\int_{\Sigma_t}2\pi^{ij}\nabla_i\beta_j d^Dx &= \int_{\Sigma_t}\left(2\pi^{ij}\nabla_i\beta^k\delta\gamma_{jk}+\pi^{ij}\beta^l\nabla_l\delta\gamma_{ij}\right)d^Dx\\
&= \int_{\Sigma_t}\pi^{k(j}\nabla_k\beta^{i)}\delta\gamma_{ij}d^Dx+\int_{\Sigma_t}\sqrt{\gamma}\nabla_l\left(\frac{1}{\sqrt{\gamma}}\pi^{ij}\beta^l\delta\gamma_{ij}\right)d^Dx\\
&\quad -\int_{\Sigma_t}\sqrt{\gamma}\nabla_l\left(\frac{1}{\sqrt{\gamma}}\pi^{ij}\beta^l\right)\delta\gamma_{ij}d^Dx\\
&= \int_{\Sigma_t}2\pi^{k(j}\nabla_k\beta^{i)}\delta\gamma_{ij}d^Dx-\int_{\Sigma_t}\nabla_l\left(\pi^{ij}\beta^l\right)\delta\gamma_{ij}d^Dx\,,
\end{aligned} \tag{B.15}$$

*where we ignore the surface term already and take the advantage of $\nabla_i\gamma\equiv 0$.*

## B.2 Couple to the matter field

**Calculation Detail B.5.** *The detail of the expression (3.7) is*

$$\begin{aligned}
\pi^q_i = M\frac{\gamma_{ij}\left(\dot{q}^j+\beta^j\right)}{\sqrt{\alpha^2-\gamma_{ij}(\dot{q}^i+\beta^i)(\dot{q}^j+\beta^j)}} &\Rightarrow (\dot{q}^i+\beta^i)\pi^q_i = M\frac{\gamma_{ij}\left(\dot{q}^j+\beta^j\right)\left(\dot{q}^i+\beta^i\right)}{\sqrt{\alpha^2-\gamma_{ij}(\dot{q}^i+\beta^i)(\dot{q}^j+\beta^j)}}\\
\Rightarrow \gamma^{ij}\pi^q_i\pi^q_j+M^2 = \frac{M^2\gamma^{ij}(\dot{q}_i+\beta_i)(\dot{q}_j+\beta_j)}{\alpha^2-\gamma_{ij}(\dot{q}^i+\beta^i)(\dot{q}^j+\beta^j)}+M^2 &= \frac{\alpha^2M^2}{\alpha^2-\gamma_{ij}(\dot{q}^i+\beta^i)(\dot{q}^j+\beta^j)}\\
\Rightarrow\quad L_M = (\dot{q}^i+\beta^i)\pi^q_i-\alpha\sqrt{\gamma^{ij}\pi^q_i\pi^q_j+M^2}&
\end{aligned} \tag{B.16}$$

**Calculation Detail B.6.** *The energy momentum tensor that describes N particle moving in the spacetime is derived as below. Consider the usual action for free particles in special relativity:*

$$S = -m \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} ds \tag{B.17}$$

*where we choose the parameter is s. If we choose it as the proper time, it is $S = -m\int ds$. If we assume that the particle trajectory is $x^\mu(s)$, we could rewrite the integral as*

$$S = -m \int d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \int ds \delta^{(n)}(x^\mu - x^\mu(s)) \tag{B.18}$$

*The Lagrangian density is*

$$\mathscr{L} = -\int ds m \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \delta^{(4)}(x^\mu - x^\mu(s)) \tag{B.19}$$

*Using the formula*

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \mathscr{L}}{\delta g_{\mu\nu}} \tag{B.20}$$

*The energy momentum tensor of a point particle actually is*

$$T^{\mu\nu}(x^\mu) = \int \frac{m}{\sqrt{-g}} \frac{d\gamma^\mu(\tau)}{d\tau} \frac{d\gamma^\nu(\tau)}{d\tau} \delta^{(D+1)}\left[x^\mu - \gamma^\mu(\tau)\right] d\tau \tag{B.21}$$

*where $\gamma^\mu(\tau)$ is the trajectory of the particle. Using the coordinate time and separate the delta function as time component and spatial component*

$$T^{\mu\nu}(t, x^i) = \int \frac{m}{\sqrt{-g}} \frac{d\gamma^\mu}{d\tau}(t') \frac{d\gamma^\nu}{d\tau}(t') \delta(t - t') \delta^{(D)}(x^i - \gamma^i(t')) \frac{d\tau}{dt}(t') dt' \tag{B.22}$$

*Finishing the integral and notice that there is a delta function,*

$$T^{\mu\nu}(t, x^i) = \frac{m}{\sqrt{-g(x^i, t)}} \frac{d\gamma^\mu}{d\tau}(t) \frac{d\gamma^\nu}{d\tau}(t) \delta^{(D)}(x^i - \gamma^i(t)) \tag{B.23}$$

*Using the condition that the particles stay at rest, the only non zero energy momentum tensor is*

$$T^{00} = \frac{m}{\sqrt{-g}} \delta^{(D)}(x^i) = \frac{m}{\sqrt{-g}} \delta^{(D)}(\vec{x}) \tag{B.24}$$

# C Properties of the Laplace operator

**Theorem C.1.** *The Green function of the operator $-\Delta + m^2$ (in the D dimensional Euclidean space) , which satisfies*

$$\left(-\Delta + m^2\right) G(\vec{x}, \vec{x}') = A\delta^{(D)}(\vec{x} - \vec{x}')\,, \tag{C.1}$$

*is*

$$G(r) = \frac{A}{(2\pi)^{D/2}} \left(\frac{m}{r}\right)^{D/2-1} K_{D/2-1}(mr)\,, \tag{C.2}$$

*where $r = |\vec{x} - \vec{x}'|$ and $K_\nu(z)$ is the modified Bessel function of the second kind.*

**Proof**: It is easy to see that the solution is rotational invariant (only relates to the relative distance $r$).

The equation reduces to

$$-\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial G}{\partial r}\right)+m^2G(r)=A\frac{\delta(r)}{\Omega_D r^{D-1}}\,, \tag{C.3}$$

where the factor $\Omega_D r^{D-1}$ comes from the volume element $dV=\Omega_D r^{D-1}dr$. Integrate both sides of the equation over a spherical volume centered at the origin with radius $r$, then take the limit as $r\to 0$, this yields

$$\lim_{r\to 0}\left[r^{D-1}\frac{\partial G}{\partial r}\right]=-\frac{A}{\Omega_D}\,. \tag{C.4}$$

The homogeneous equation of $G$ is

$$-\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial G}{\partial r}\right)+m^2G(r)=0\,. \tag{C.5}$$

If we set $G(r)=g(r)r^{-(D/2-1)}$, the equation becomes

$$r^2\frac{\partial^2 f}{\partial r^2}+r\frac{\partial f}{\partial r}-\left(\frac{D}{2}-1\right)^2 f-m^2r^2f=0\,. \tag{C.6}$$

This is a standard ODE and its solution is

$$f(r)=CK_{D/2-1}(mr)+DI_{D/2-1}(mr)\,, \tag{C.7}$$

where $I_{D/2-1}$ and $K_{D/2-1}$ are modified Bessel functions of the first and second kind and $C,D$ are constants determined by boundary conditions. A natural boundary is that $\lim_{r\to\infty}G=0$, which requires that $D=0$. So

$$G(r)=\frac{C}{r^{D/2-1}}K_{D/2-1}(mr)\,. \tag{C.8}$$

Using the condition when $r\to 0$, we have

$$\lim_{r\to 0}\left[r^{D-1}\frac{\partial G}{\partial r}\right]=-\Gamma\left(\frac{D}{2}\right)2^{\frac{D}{2}-1}m^{1-\frac{D}{2}}C=-\frac{A}{\Omega_D}\,, \tag{C.9}$$

where we use the approximate form of $K_{D/2-1}(mr)$ near $r=0$. This implies

$$C=\frac{Am^{D/2-1}}{2^{D/2-1}\Gamma(D/2)\Omega_D}\,, \tag{C.10}$$

with $\Omega_d=2\pi^{D/2}/\Gamma(D/2)$ we have

$$C=\frac{Am^{D/2-1}}{2^{D/2}\pi^{D/2}}\,. \tag{C.11}$$

The solution is

$$G(r)=\frac{A}{(2\pi)^{D/2}}\left(\frac{m}{r}\right)^{\frac{D}{2}-1}K_{D/2-1}(mr)\,. \tag{C.12}$$

**Theorem C.2.** *The Green's function of the operator $-\Delta$, which satisfies*

$$-\Delta G=A\delta^D(\vec{x}-\vec{x}')\,, \tag{C.13}$$

*is*

$$G(r) = \lim_{m\to 0} G_m(r) = \frac{A}{(2\pi)^{D/2}} \lim_{m\to 0} \left(\frac{m}{r}\right)^{D/2-1} \frac{\Gamma(D/2-1)}{2} \left(\frac{2}{mr}\right)^{D/2-1} = \frac{A\Gamma\left(\frac{D}{2}-1\right)}{4\pi^D r^{D-2}} \tag{C.14}$$

**Proof:** This is direct since we already had the general Green function for operator $-\Delta + m^2$.

# D Proof of the solution

To verify it, we substitute it into the equation 4.13. First

$$\begin{aligned} \frac{1}{r^{D-1}}\frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} f &= \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \left[\frac{-2}{D-2} f^{D/2} \frac{\partial}{\partial r}\left(1 + \frac{r_0^{D-2}}{r^{D-2}}\right)\right] \\ &= \frac{-2}{D-2} \frac{\partial f^{D/2}}{\partial r} \frac{\partial f^{-\frac{D-2}{2}}}{\partial r} + \frac{-2}{D-2} f^{D/2} \left(-\frac{4\pi^D r_0^{D-2}}{\Gamma(D/2-1)} \delta^{(D)}(\vec{r})\right) \end{aligned} \tag{D.1}$$

$$\Rightarrow \quad 2f \frac{1}{r^{D-1}}\frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} f = D\left(\frac{\partial f}{\partial r}\right)^2 + \frac{16\pi^D r_0^{D-2}}{\Gamma(D/2-1)} f^{D/2+1} \delta^{(D)}(\vec{r}) \,, \tag{D.2}$$

where the first term cancels the first derivative part exactly. What we left is the term

$$\frac{16\pi^D (D-1) r_0^{D-2}}{\Gamma(D/2-1)} f^{1-D/2} \delta^{(D)}(\vec{r}) \,. \tag{D.3}$$

At first glance it is not equal to the delta function in the right hand side (the factor $f^{1-D/2}$). Let's consider it carefully. Using the specific form of $f$, we obtain

$$\frac{16\pi^D (D-1) r_0^{D-2}}{\Gamma(D/2-1)} \left(1 + \frac{r_0^{D-2}}{r^{D-2}}\right) \delta^{(D)}(\vec{r}) \,. \tag{D.4}$$

The unexpected term is

$$\frac{\delta^{(D)}(\vec{r})}{r^{D-2}} \,. \tag{D.5}$$

Recall that we should define the delta function as a functional. Inner product with arbitrary function $\phi : \mathbb{R}^D \to \mathbb{R}$ is

$$\left(\frac{1}{r^{D-2}} \delta^{(D)}(\vec{r}), \phi\right) \,. \tag{D.6}$$

The expression $\delta^{(D)}(\vec{r})/r^{D-2}$ is proportional to

$$\frac{1}{r^{D-2}} \Delta \frac{1}{r^{D-2}} \,. \tag{D.7}$$

Let's consider

$$\Delta \frac{r^\lambda}{r^{D-2}} = \left[\lambda^2 + \lambda(2-D)\right] r^{\lambda - D} + r^\lambda \Delta \frac{1}{r^{D-2}} \,. \tag{D.8}$$

Analytically continued to $\lambda = -(D-2)$, we could rewrite the expression

$$\frac{1}{r^{D-2}} \Delta \frac{1}{r^{D-2}} = \Delta \frac{1}{r^{2D-4}} - 2(D-2)^2 r^{-2D+2} = 2(2-D)^2 \frac{1}{r^{2D-2}} - 2(D-2)^2 r^{-2D+2} \equiv 0 \,. \tag{D.9}$$

So we indeed find the solution $f$ for the equation (4.10).